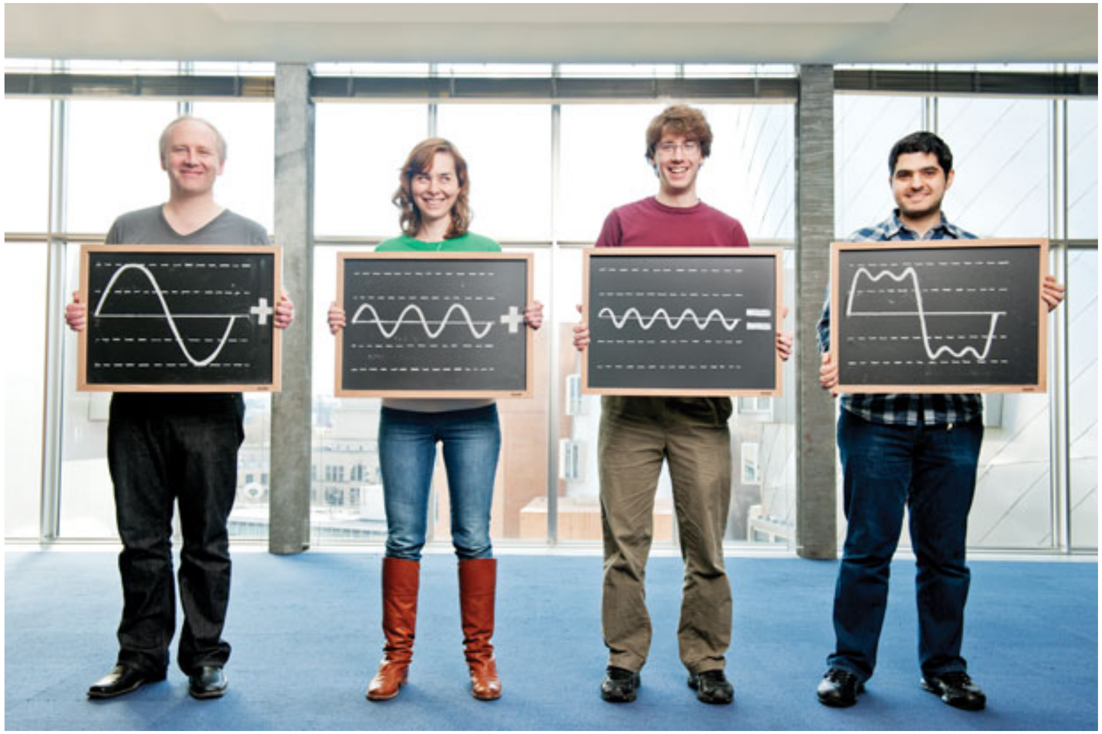
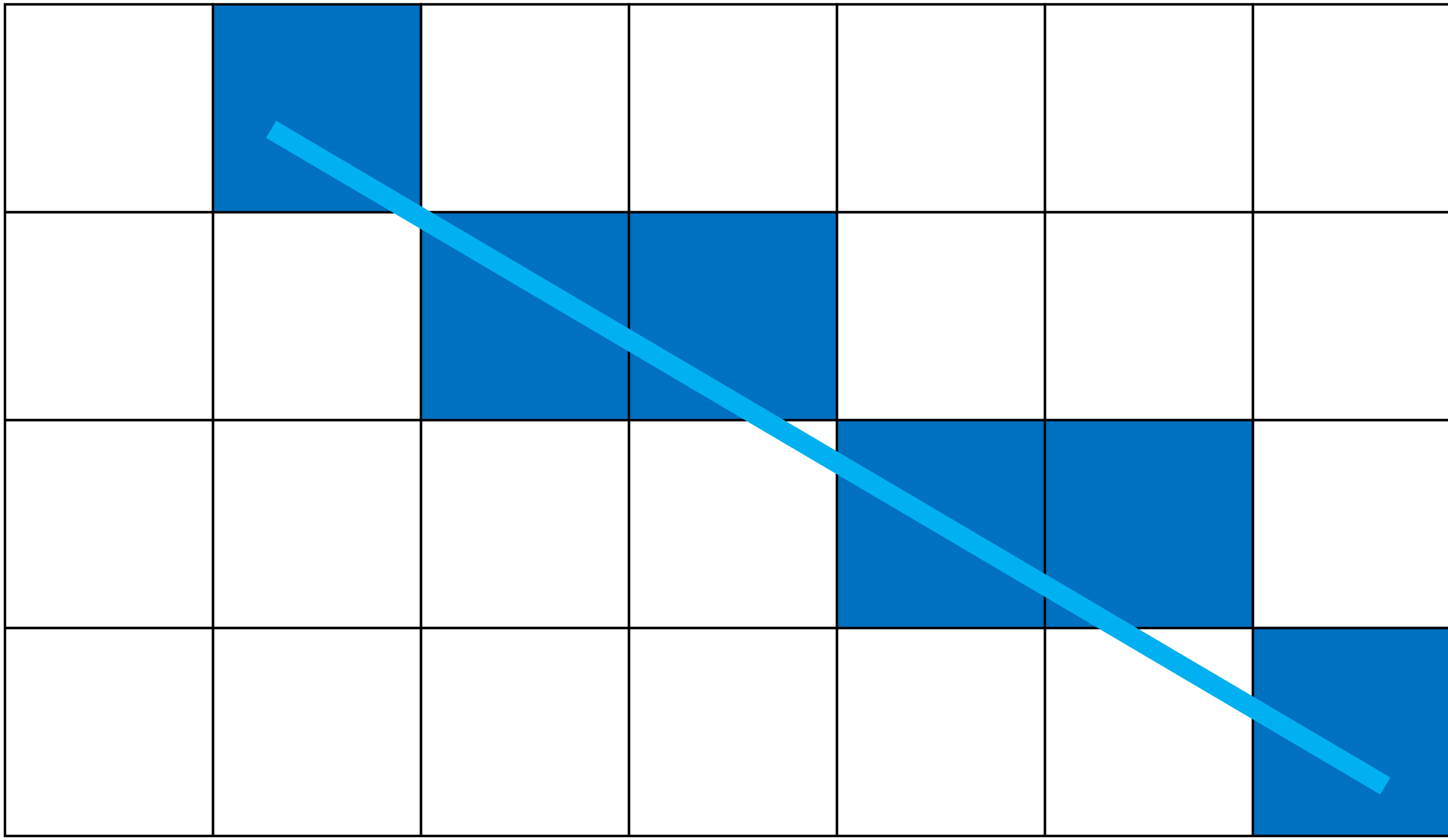


Sampling

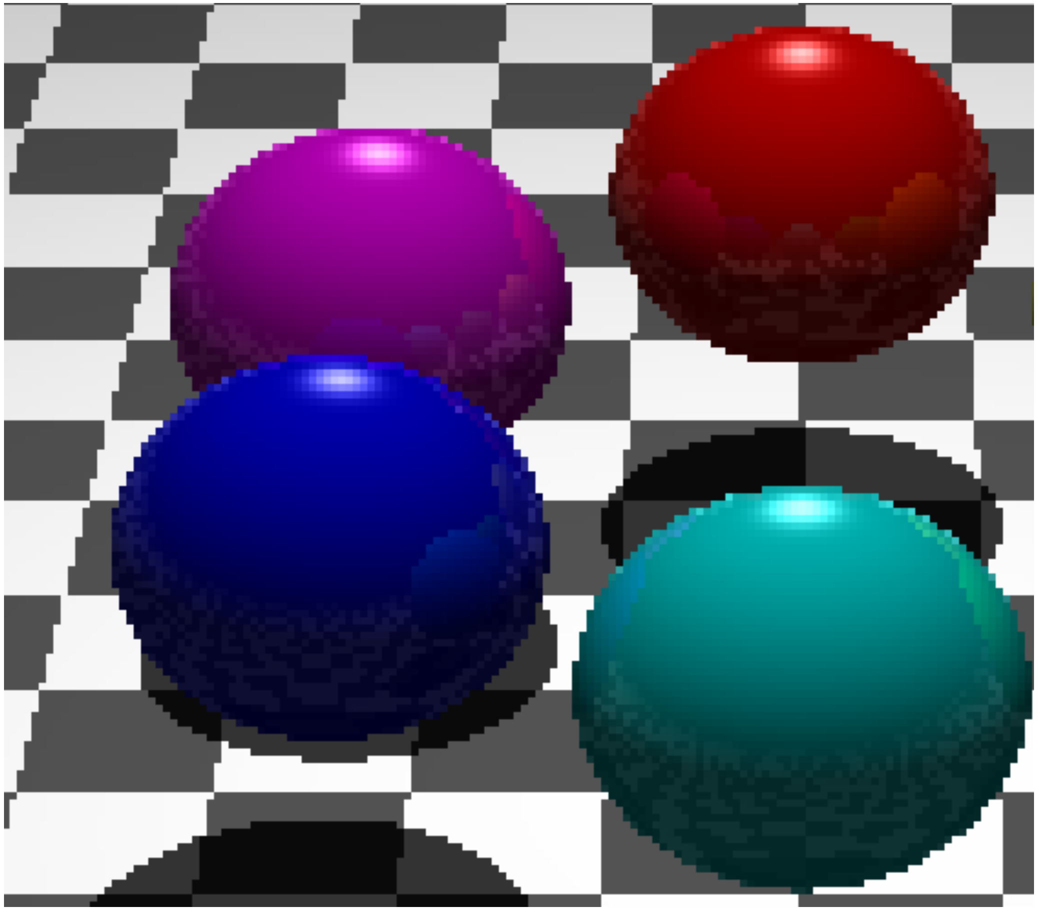
CS 148: Summer 2016
Introduction of Graphics and Imaging
Zahid Hossain

Important Issues We've Seen



Jaggies

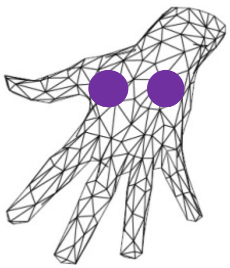
Important Issues We've Seen



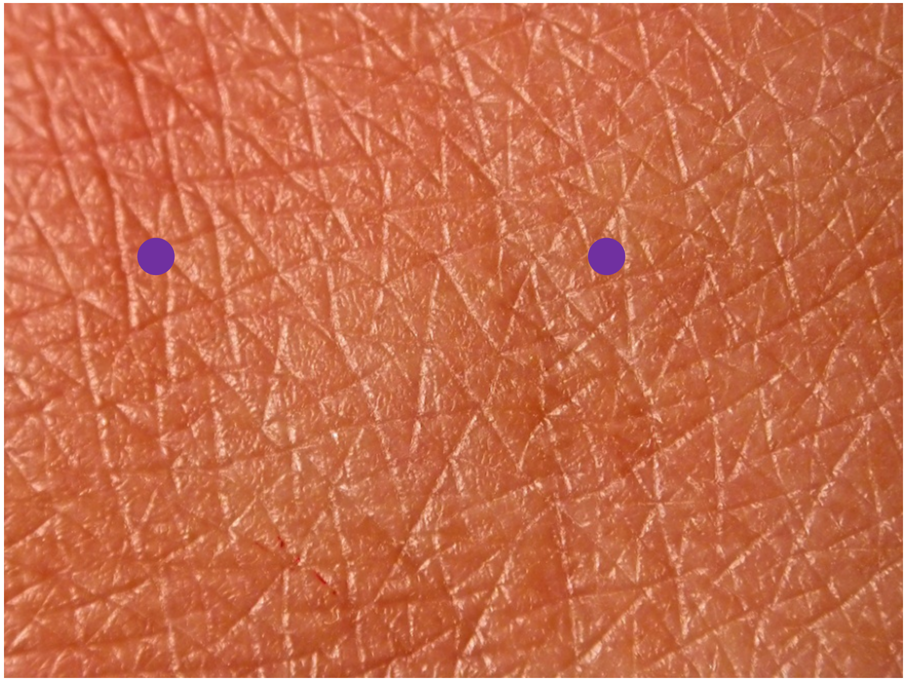
More Jaggies

Important Issues We've Seen

Small image

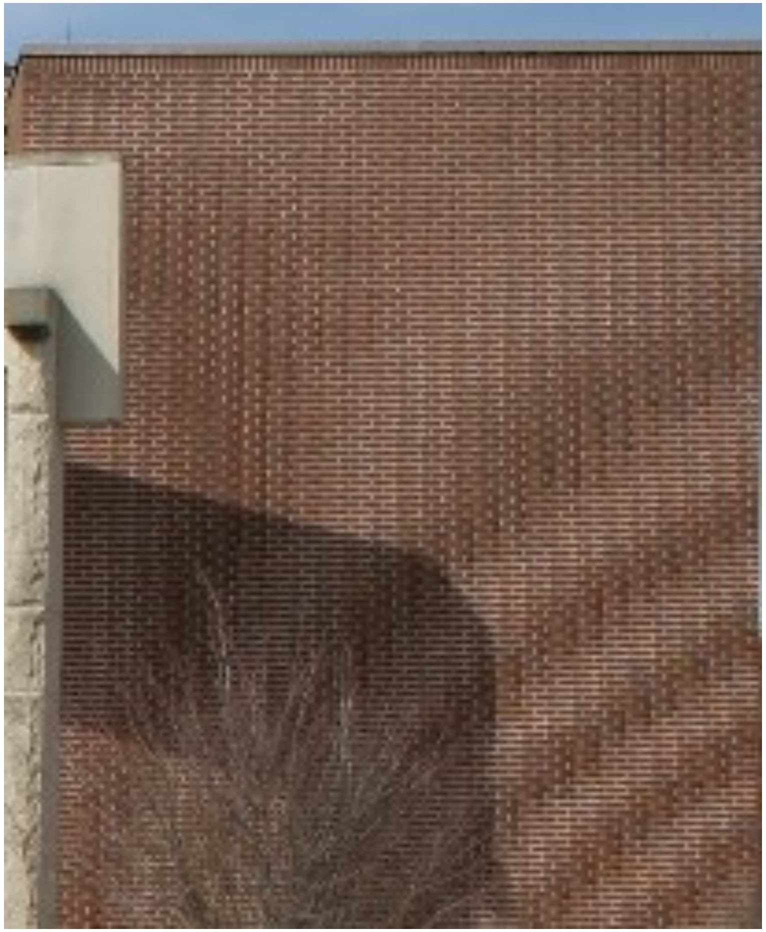


Large texture



Magnification/Minification

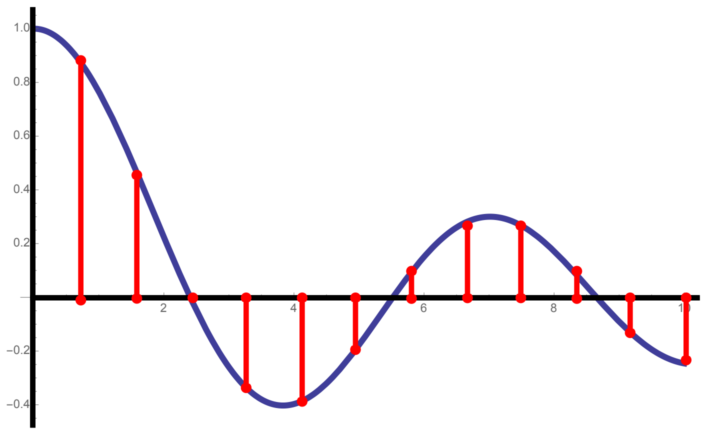
Important Issues We've Seen



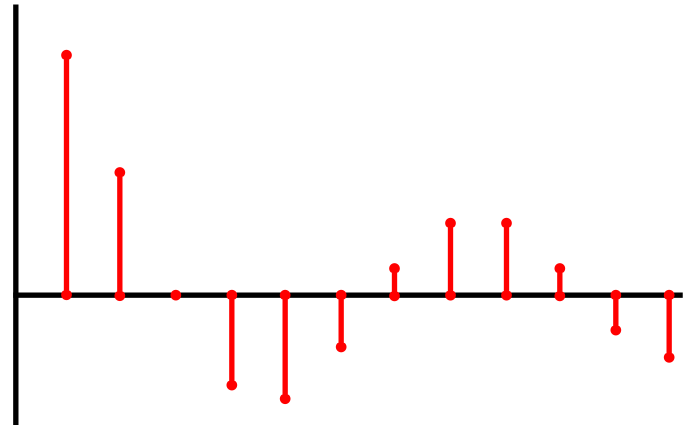
Moiré Patterns

What’s Broken ?

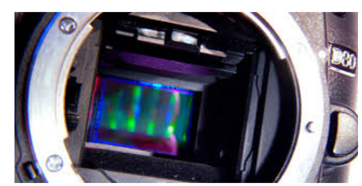
World is Continuous



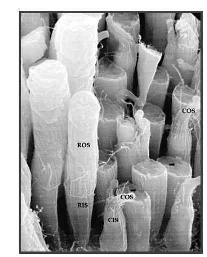
Storage, Sensor, Display are Discrete



RAM

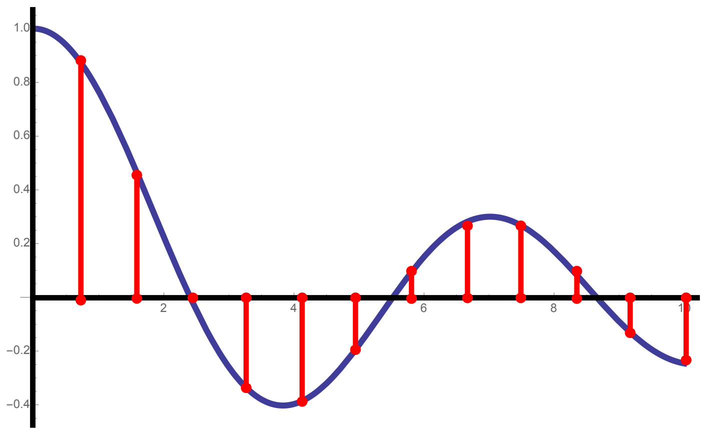


Camera Sensor

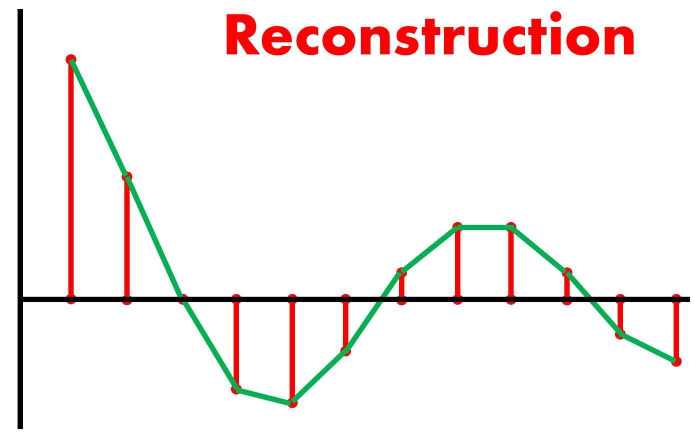


Eye Retina Cells

What’s Broken ?

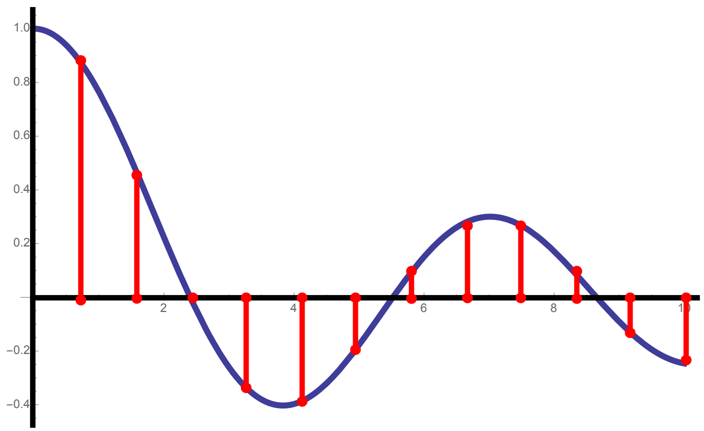


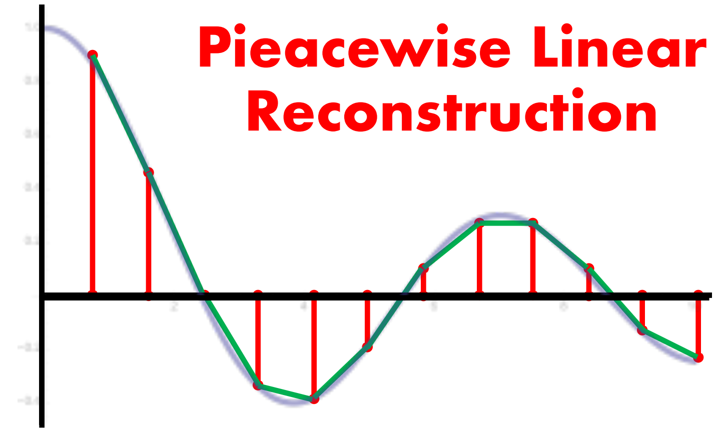
Pieacewise Linear Reconstruction



Reconstruct from samples

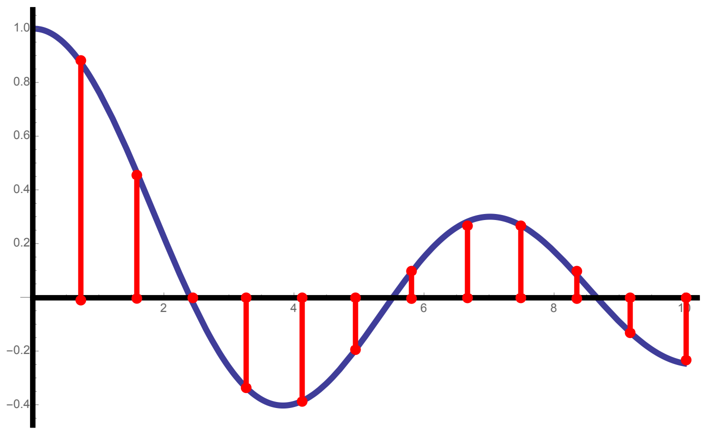
What’s Broken ?

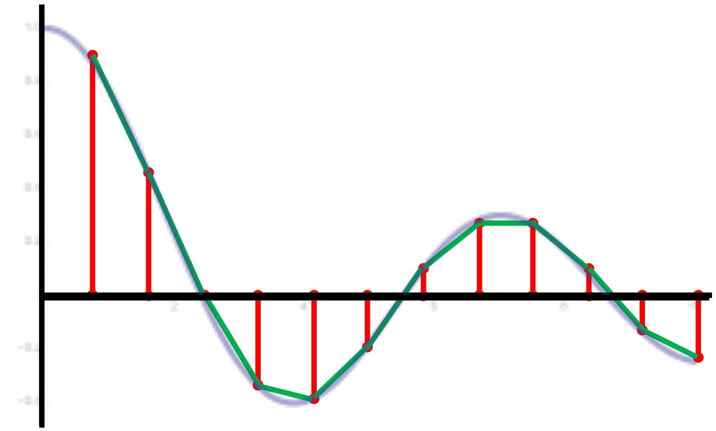


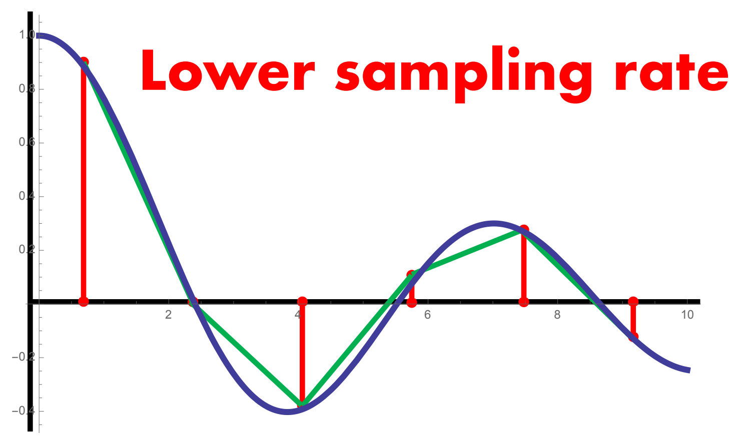


May work well with lots of samples

What's Broken ?

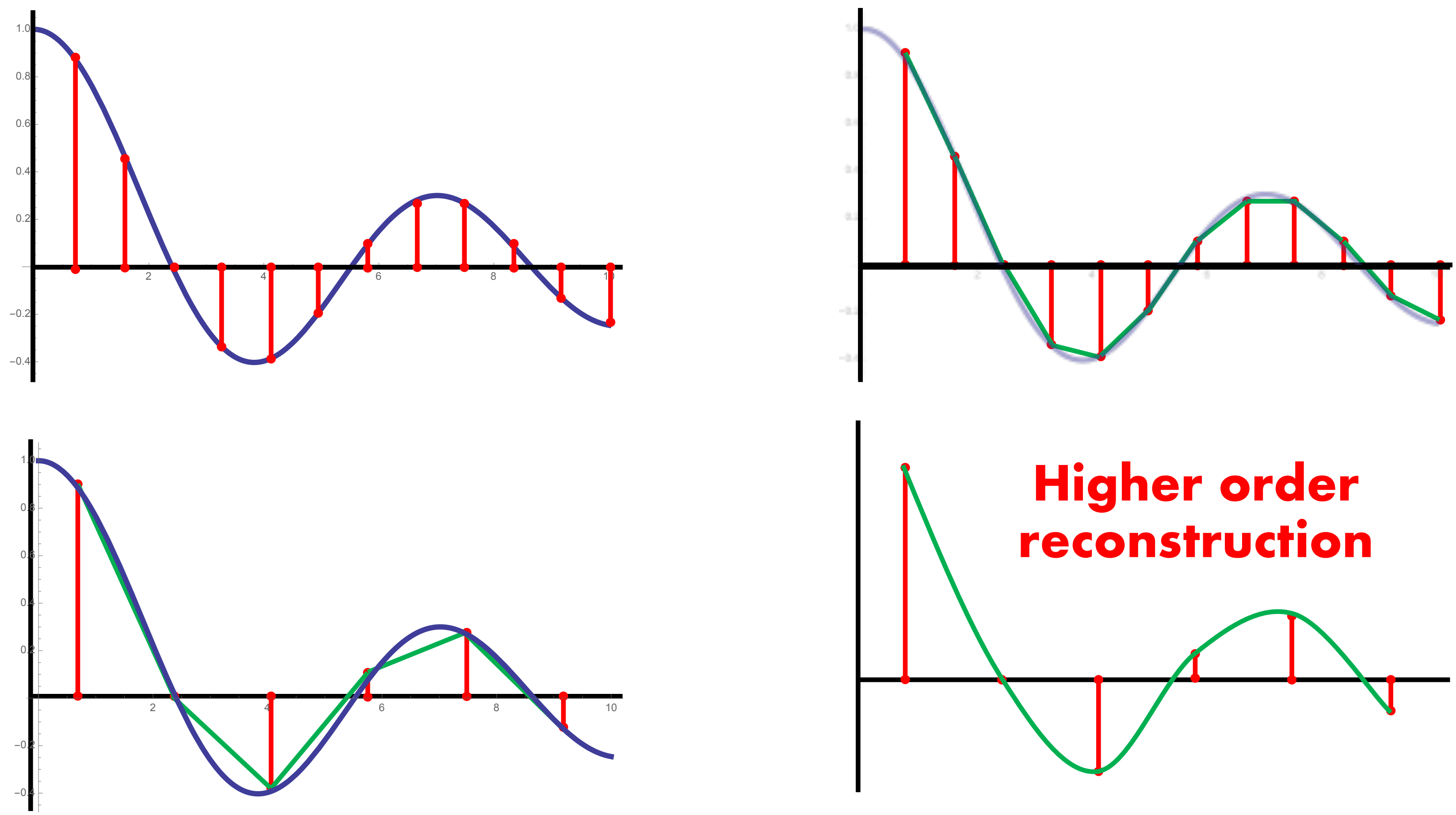




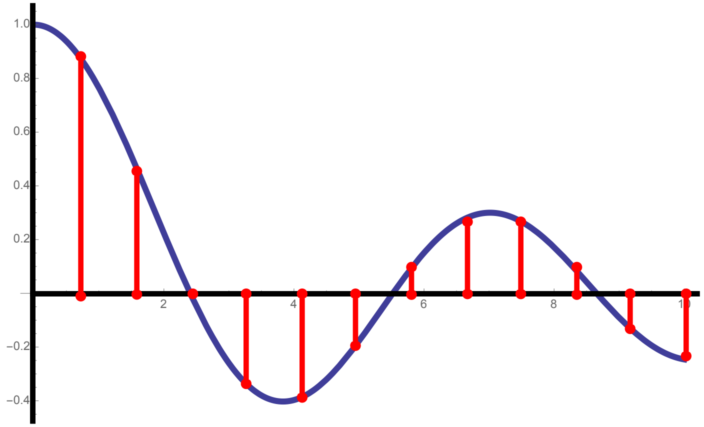


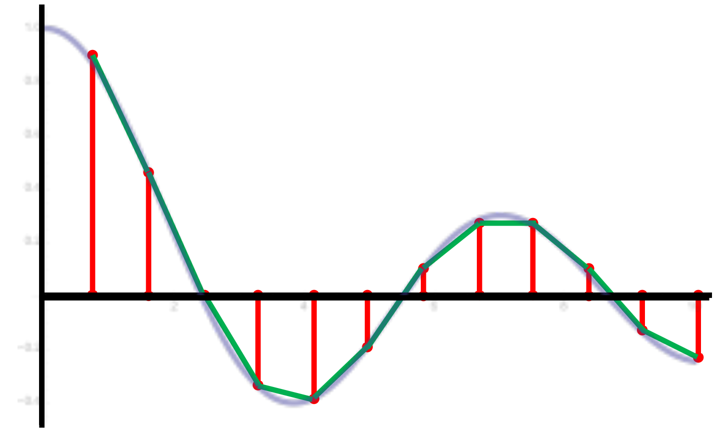
Not so much with less samples

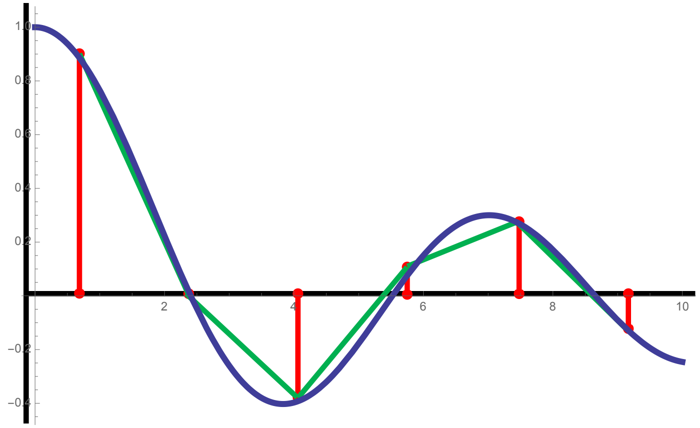
What's Broken ?

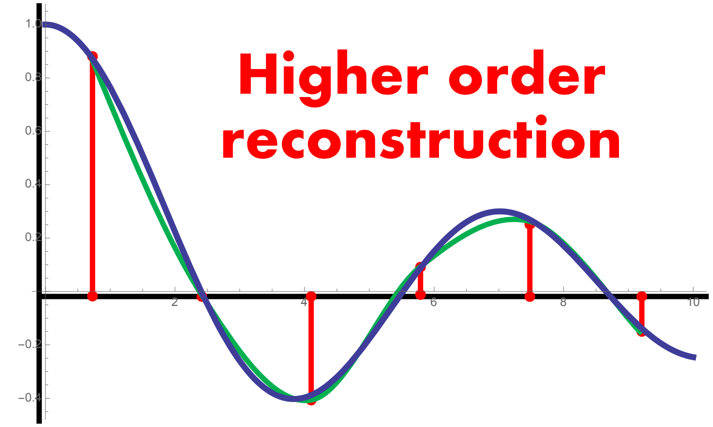


What's Broken ?



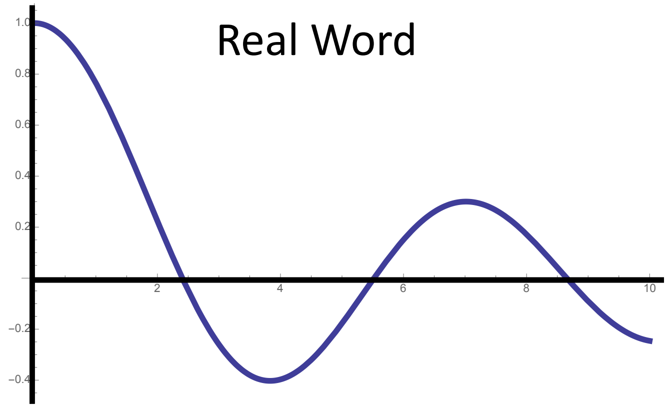


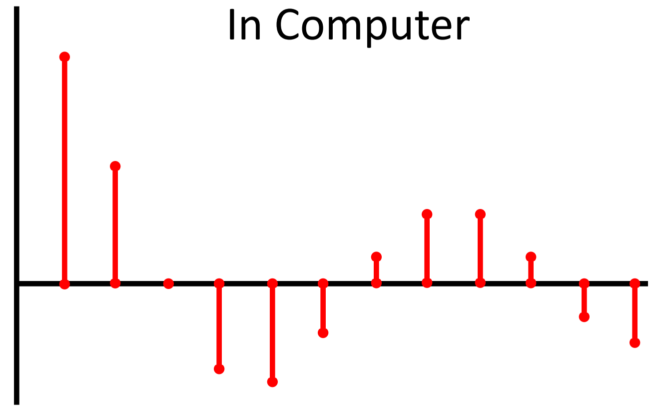




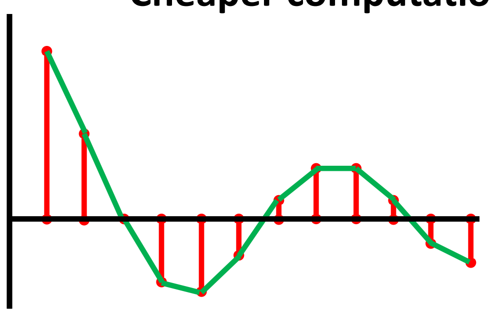
With better reconstruction: we do better

Tradeoffs

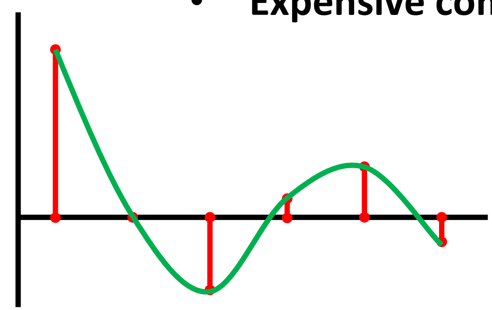




- **Higher sampling rate**
  - **Large storage space**
- **Get away with lower order reconstruction**
  - **Cheaper computation**



- **Lower sampling rate**
  - **Smaller storage space**
- **Requires higher order reconstruction**
  - **Expensive computation**



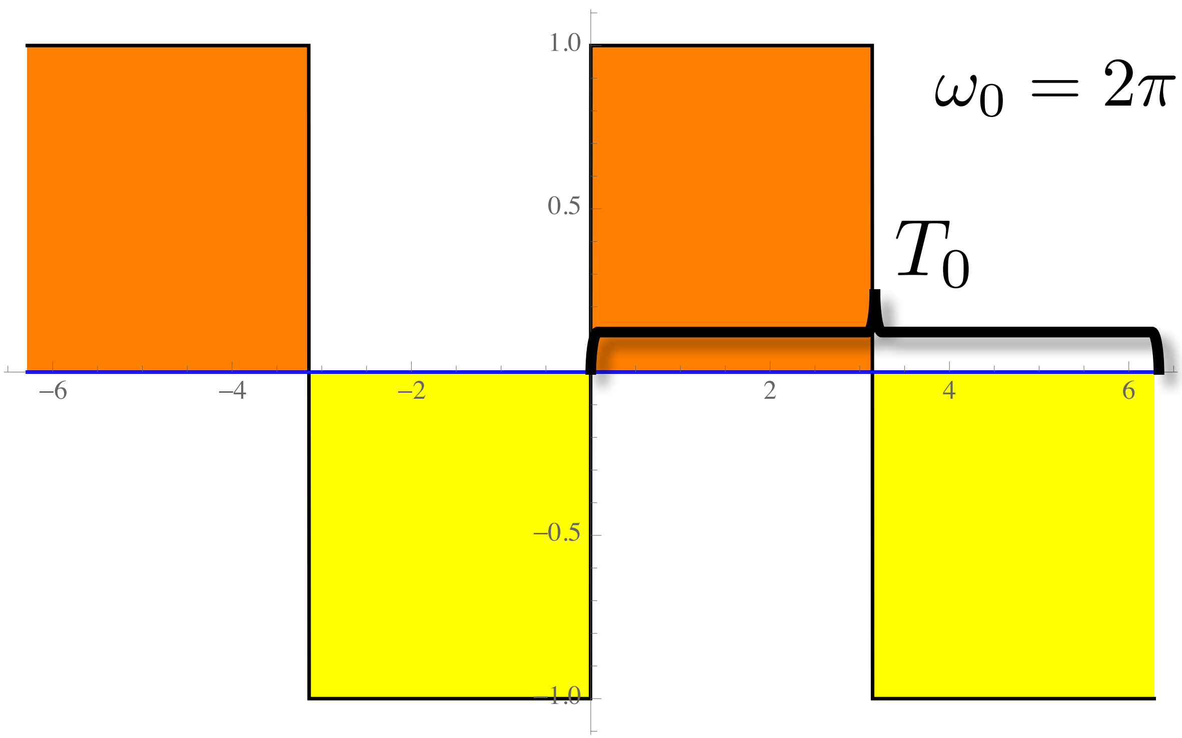
Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$

Fourier Series

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$



Fourier Series

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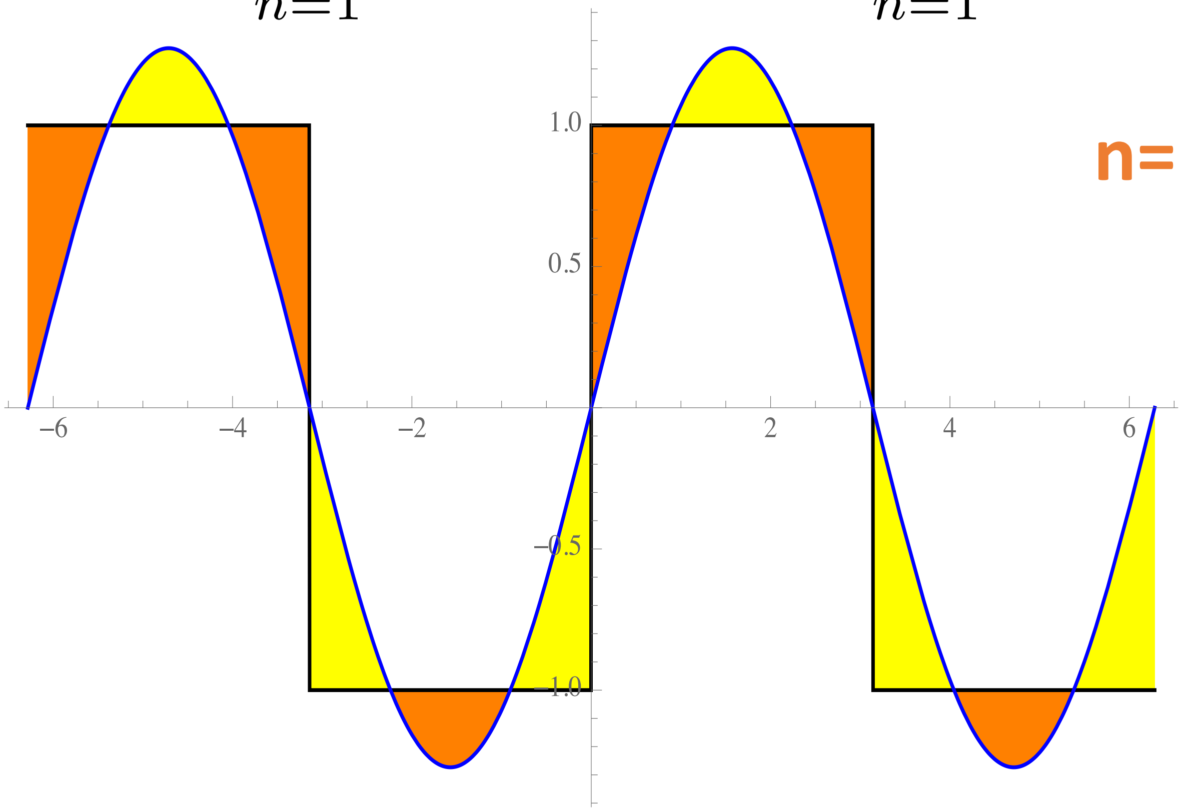
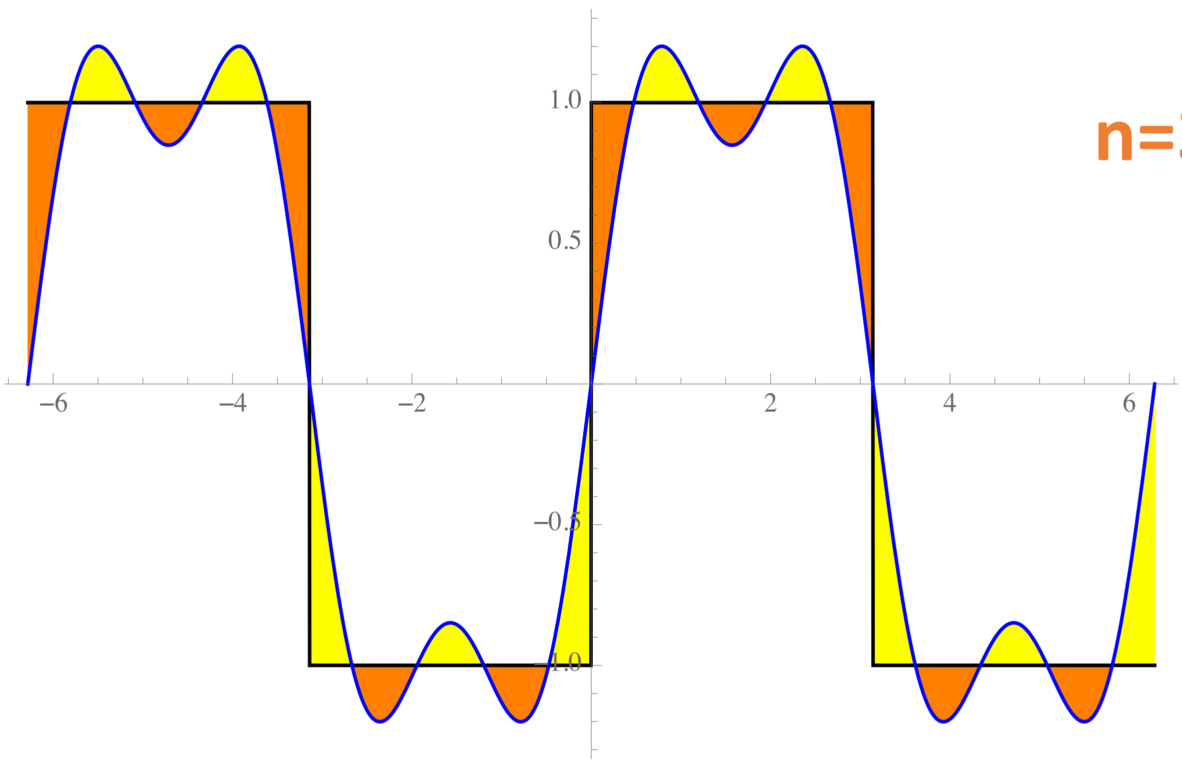


n=1

Fourier Series

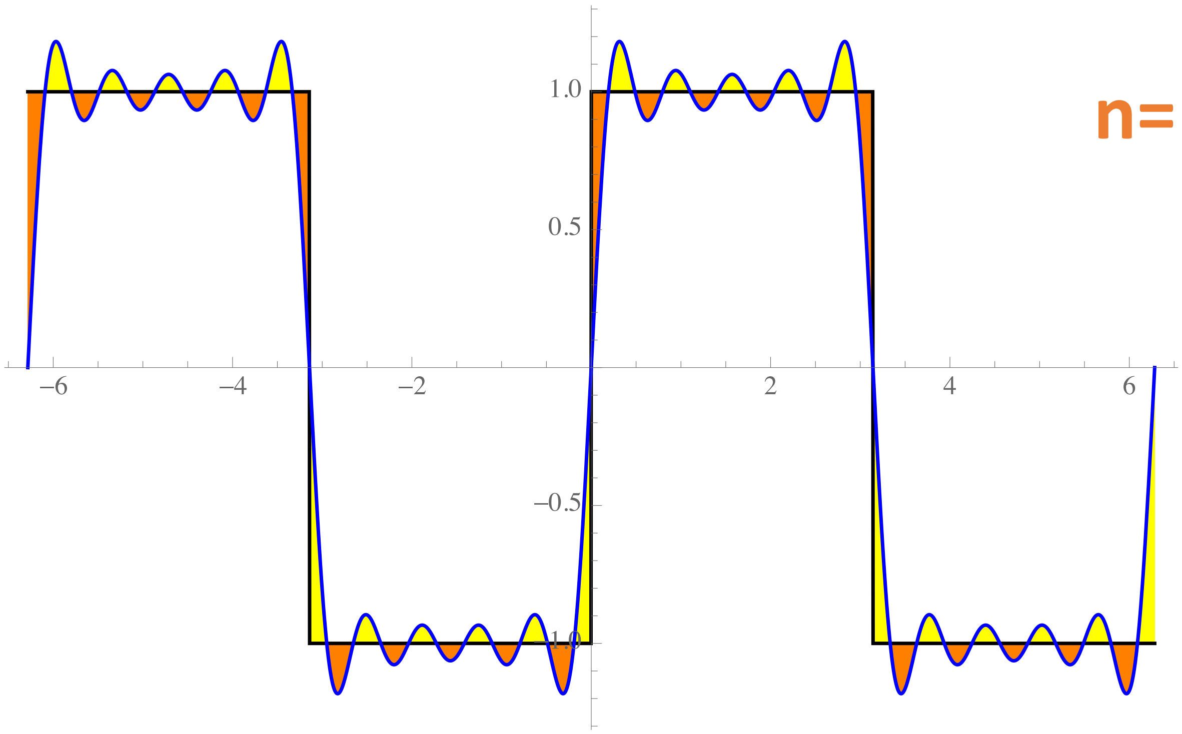
$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$



n=3

Fourier Series

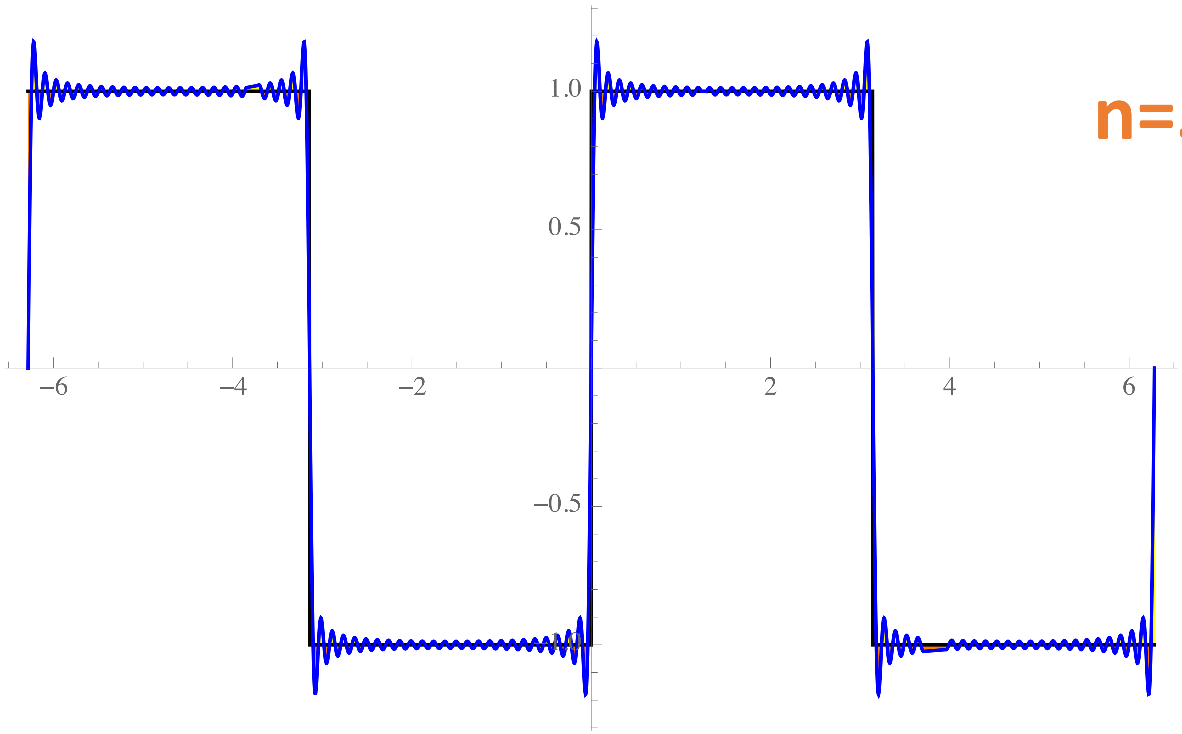
$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$



n=10

Fourier Series

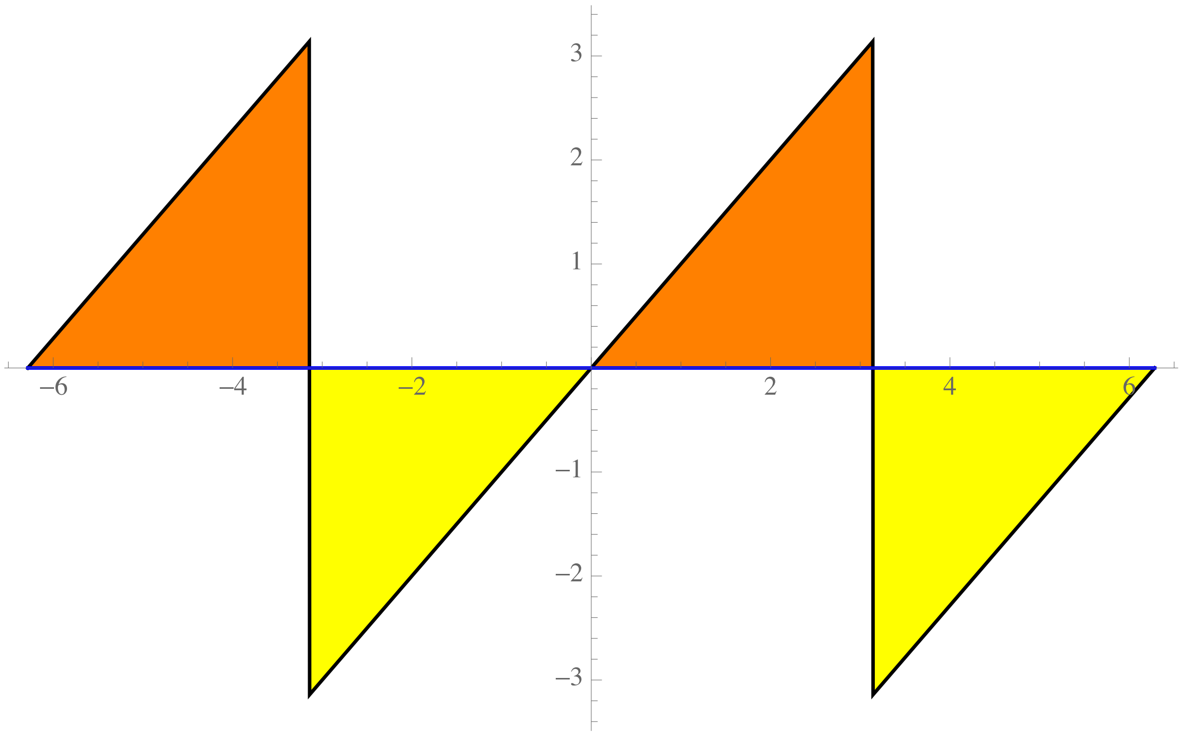
$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$



n=50

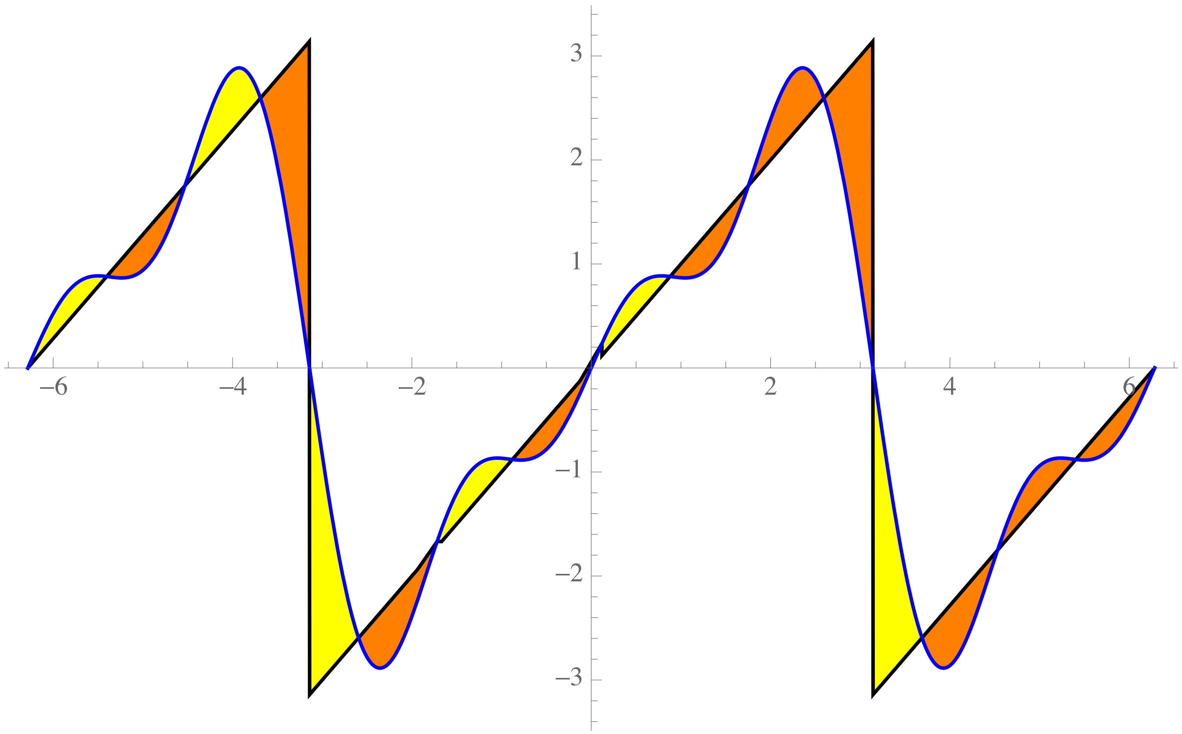
Fourier Series: More Examples

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$



Fourier Series: More Examples

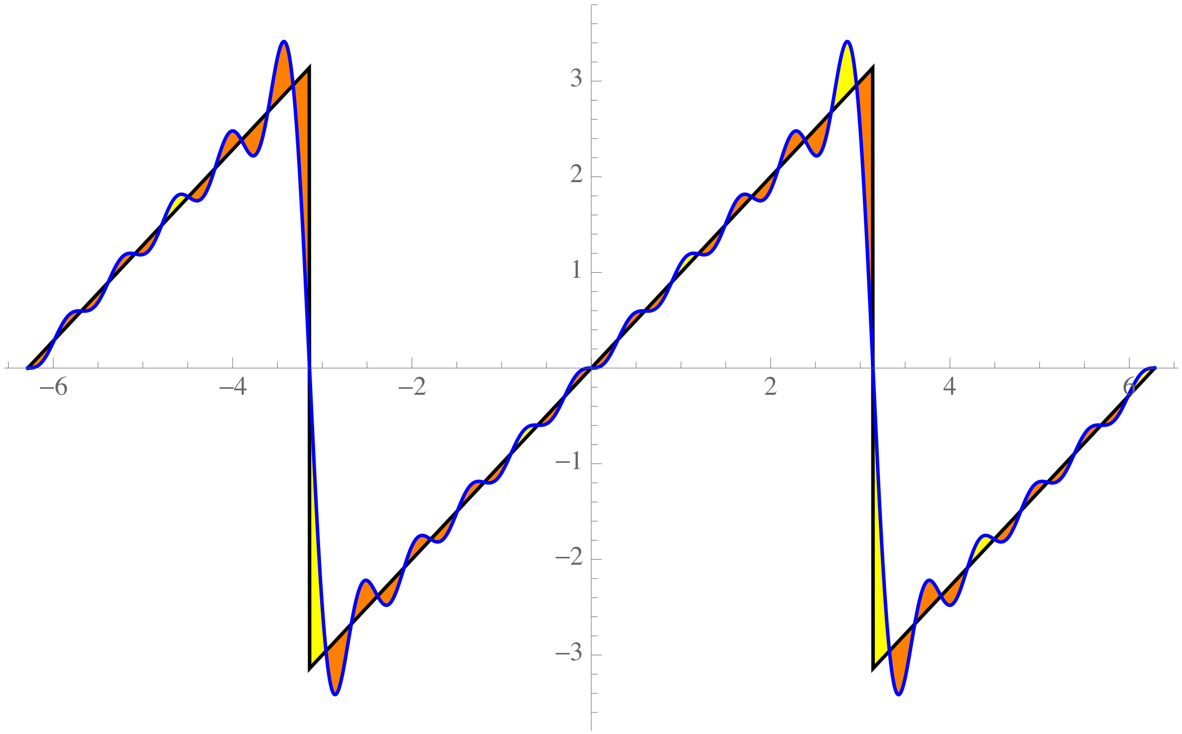
$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$



n=3

Fourier Series: More Examples

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$



n=10

Computing the Coefficients

- Treat functions as vectors !
- Define a inner product (dot product)

$$\text{Inner Product} : \langle g, h \rangle \stackrel{def}{\equiv} \int_x g(x)\overline{h(x)}dx$$

Computing the Coefficients

- Treat functions as vectors !
- Define a inner product (dot product)
- Different frequencies of Cos and Sin happens to be orthogonal to each other

$$\langle \sin(n\omega_0 x), \sin(m\omega_0 x) \rangle = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin(n\omega_0 x) \sin(m\omega_0 x) dx$$

$$= \begin{cases} \frac{1}{2} T_0 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

Computing the Coefficients

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$$\langle \cos(n\omega_0 x), \cos(m\omega_0 x) \rangle = \begin{cases} \frac{1}{2}T_0 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \sin(n\omega_0 x), \cos(m\omega_0 x) \rangle = 0$$

Computing the Coefficients

$$\text{Orthogonal set} : \{1, \sin(n\omega_0 x), \cos(m\omega_0 x)\}$$

$$
\begin{aligned}
a_0 &= 2\tfrac{<g(x),1>}{<1,1>} &&= \tfrac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x)dx \\
a_n &= \tfrac{\langle g(x),\cos(n\omega_0 x)\rangle}{\langle \cos(n\omega_0 x),\cos(n\omega_0 x)\rangle} &&= \tfrac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x)\cos(n\omega_0 x)dx \\
b_n &= \tfrac{\langle g(x),\sin(n\omega_0 x)\rangle}{\langle \sin(n\omega_0 x),\sin(n\omega_0 x)\rangle} &&= \tfrac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x)\sin(n\omega_0 x)dx
\end{aligned}
$$

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$

Fourier Series: Compact Form

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$

Compact Form

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$

Fourier Series: Compact Form

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o x)$$

Compact Form

$$e^{in\omega_0 x} = \cos(n\omega_0 x) + i\sin(in\omega_0 x)$$

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$

Fourier Series: Compact Form

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$

$$\text{Orthogonal set} : \{e^{in\omega_0 x}\}$$

$$\begin{aligned}\left\langle e^{in\omega_0 x}, e^{im\omega_0 x}\right\rangle &= \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{in\omega_0 x} e^{-im\omega_0 x} dx \\ &= \begin{cases} T & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Fourier Series: Compact Form

$$g(x) = \sum^{\infty}$$

Note the complex conjugate in the inner product definition

set : $\{e^{in\omega_0 x}\}$

$$\langle e^{in\omega_0 x}, e^{im\omega_0 x} \rangle = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{in\omega_0 x} e^{-im\omega_0 x} dx$$

$$= \begin{cases} T & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

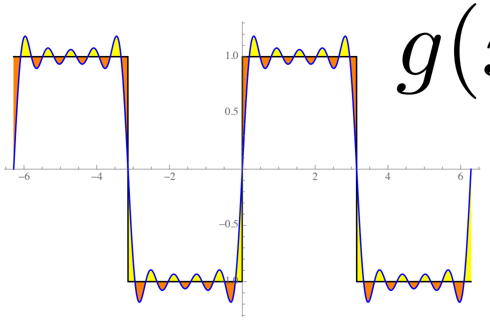
Fourier Series: Coefficients

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$

$$A_n = \frac{\langle g(x), e^{in\omega_0 x} \rangle}{\langle e^{in\omega_0 x}, e^{in\omega_0 x} \rangle} = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-in\omega_0 x} dx$$

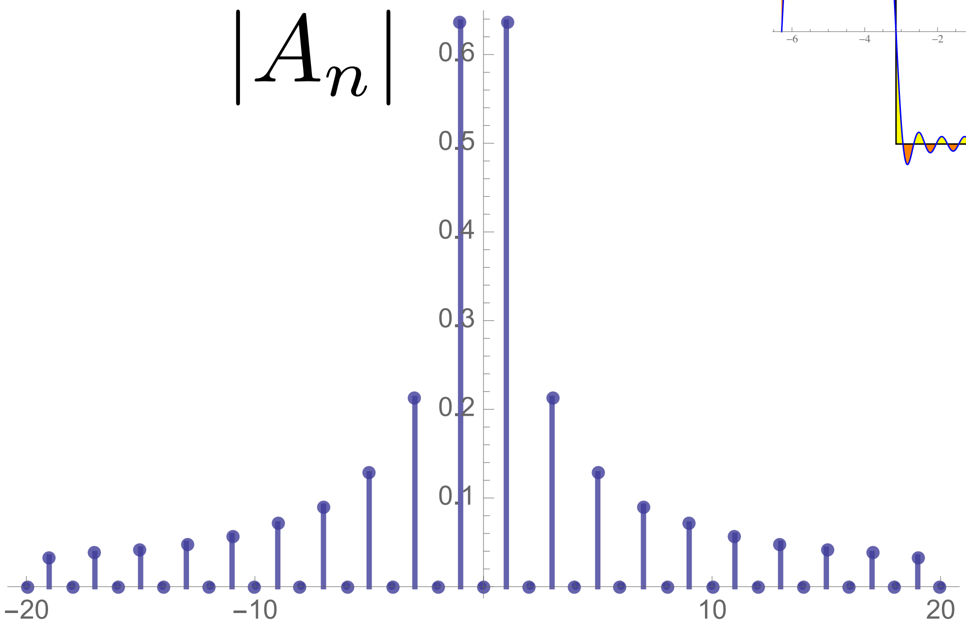
Distribution of Frequencies

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$



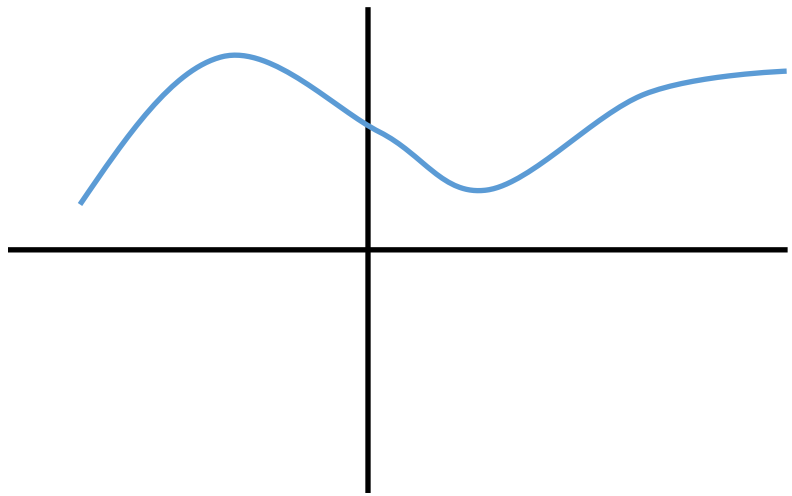
$g(x)$

$|A_n|$



n

Fourier Transform: Continuous



$$T_0 \to \infty$$

$$\lim_{T_0 \to \infty} \omega_0 = \frac{2\pi}{T_0} = 0$$

Fourier Transform: Continuous

$$T_0 \to \infty \qquad \lim_{T_0 \to \infty} \omega_0 = \frac{2\pi}{T_0} = 0$$

$$A_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-in\omega_0 x} dx$$

Fourier Transform: Continuous

$$T_0 \to \infty \qquad \lim_{T_0 \to \infty} \omega_0 = \frac{2\pi}{T_0} = 0$$

$$A_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-in\omega_0 x} dx$$

$$\implies A_n = \frac{\omega_0}{2\pi} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-i\omega x} dx, \qquad \left[\omega = n\omega_0, \omega_0 = \frac{2\pi}{T_0}\right]$$

Fourier Transform: Continuous

$$T_0 \to \infty \qquad \lim_{T_0 \to \infty} \omega_0 = \frac{2\pi}{T_0} = 0$$

$$A_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-in\omega_0 x} dx$$

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$$\implies \lim_{T_0 \to \infty, \omega_0 \to 0} A_n = \lim_{T_0 \to \infty, \omega_0 \to 0} \frac{\omega_0}{2\pi} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-i\omega x} dx$$

Fourier Transform: Continuous

$$T_0 \to \infty \qquad \lim_{T_0 \to \infty} \omega_0 = \frac{2\pi}{T_0} = 0$$

$$A_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-in\omega_0 x} dx$$

$$\implies A_n = \frac{\omega_0}{2\pi} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-i\omega x} dx, \qquad \left[\omega = n\omega_0, \omega_0 = \frac{2\pi}{T_0}\right]$$

$$\implies \lim_{T_0 \to \infty, \omega_0 \to 0} A_n = \lim_{T_0 \to \infty, \omega_0 \to 0} \frac{\omega_0}{2\pi} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g(x) e^{-i\omega x} dx$$

$$\implies \boxed{\lim_{T_0 \to \infty, \omega_0 \to 0} A_n = \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx, \qquad [\omega_0 = \Delta\omega]}$$

Fourier Transform: Continuous

$$\lim_{T_0 \to \infty} A_n = \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$$

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$

Fourier Transform: Continuous

$$\lim_{T_0 \to \infty} A_n = \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$$

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$

$$g(x) = \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \right) e^{i\omega x}, \quad [\omega = n\omega_0]$$

Fourier Transform: Continuous

$$\lim_{T_0 \to \infty} A_n = \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$$

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$

$$\begin{aligned} g(x) &= \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \right) e^{i\omega x}, \quad [\omega = n\omega_0] \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \right) e^{i\omega x} \Delta\omega \end{aligned}$$

Fourier Transform: Continuous

$$\lim_{T_0 \to \infty} A_n = \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$$

$$g(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 x}$$

$$g(x) = \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \right) e^{i\omega x}, \quad [\omega = n\omega_0]$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \right) e^{i\omega x} \Delta\omega$$

$$\lim_{\omega_0 \to 0} g(x) = \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \right)}_{F(\omega)} e^{i\omega x} d\omega$$

Fourier Transform: Continuous

Fourier Transform of the function $g(x)$

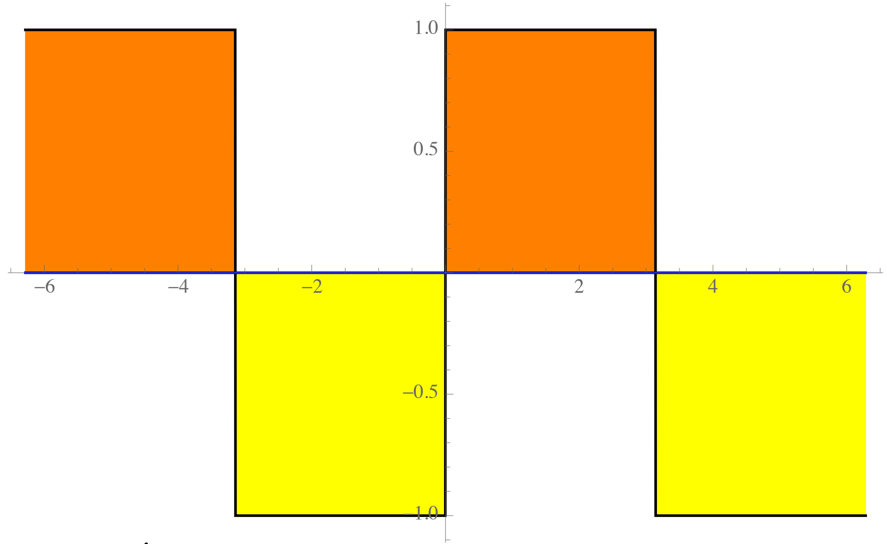
$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$$

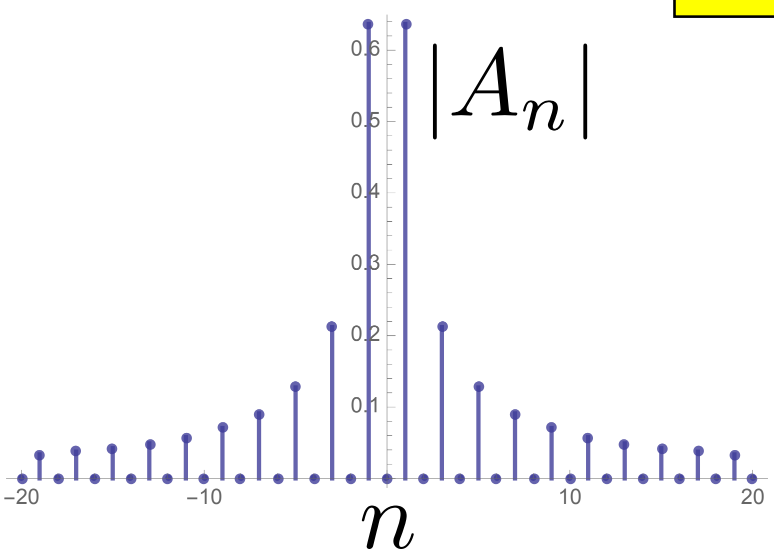
Fourier Transform: Operator Form

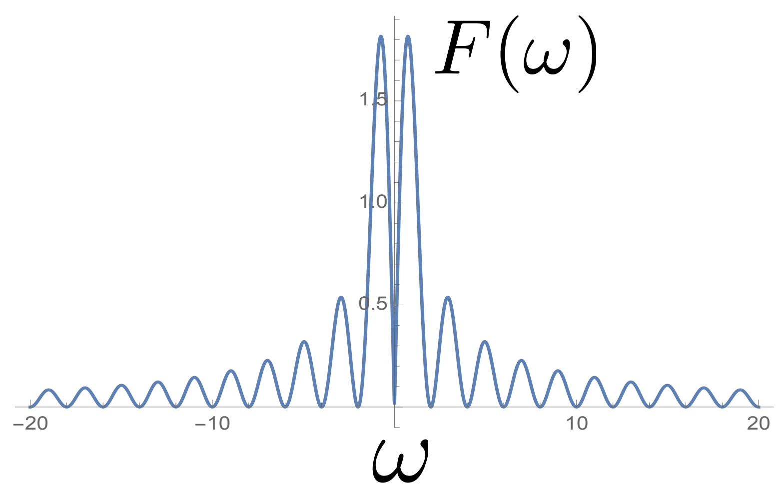
$$\mathcal{F}\left[g(x)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx$$

$$\mathcal{F}^{-1}\left[F(\omega)\right] = g(x) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega$$

Fourier Transform: Examples



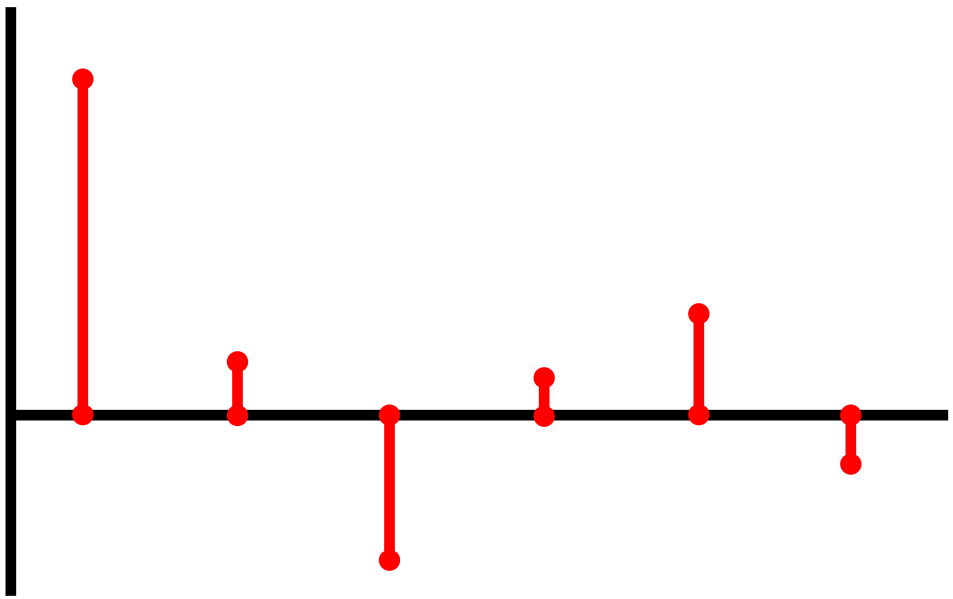




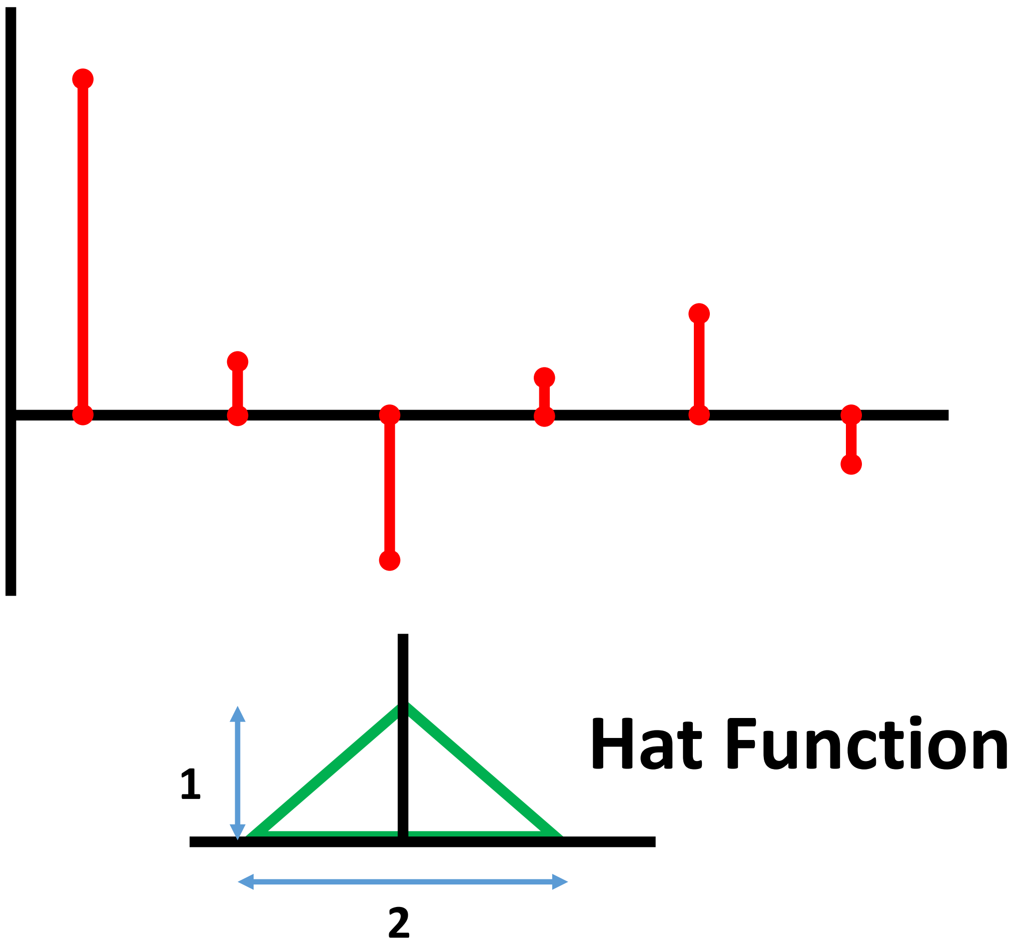
Fourier Transform Properties

- Fourier transform is a linear operator
  - $F[\ ag(x) + bh(x)] = a\ F[\ g(x)\ ] + b\ F[\ h(x)]$
- Fourier transform scaling property
  - $f(\lambda x) \Leftrightarrow \frac{1}{|\lambda|} F\left(\frac{\omega}{\lambda}\right)$

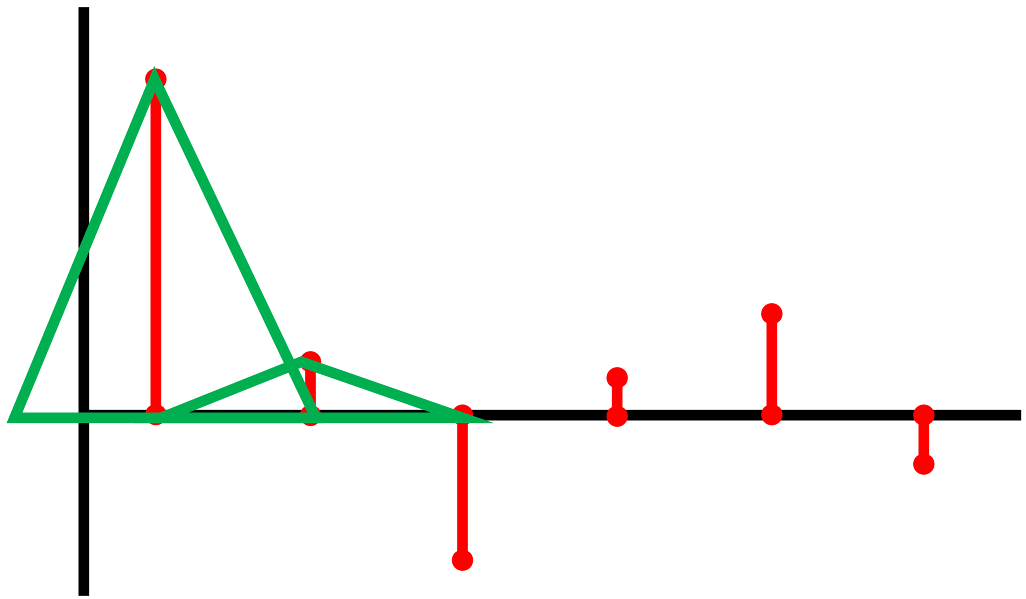
Convolution



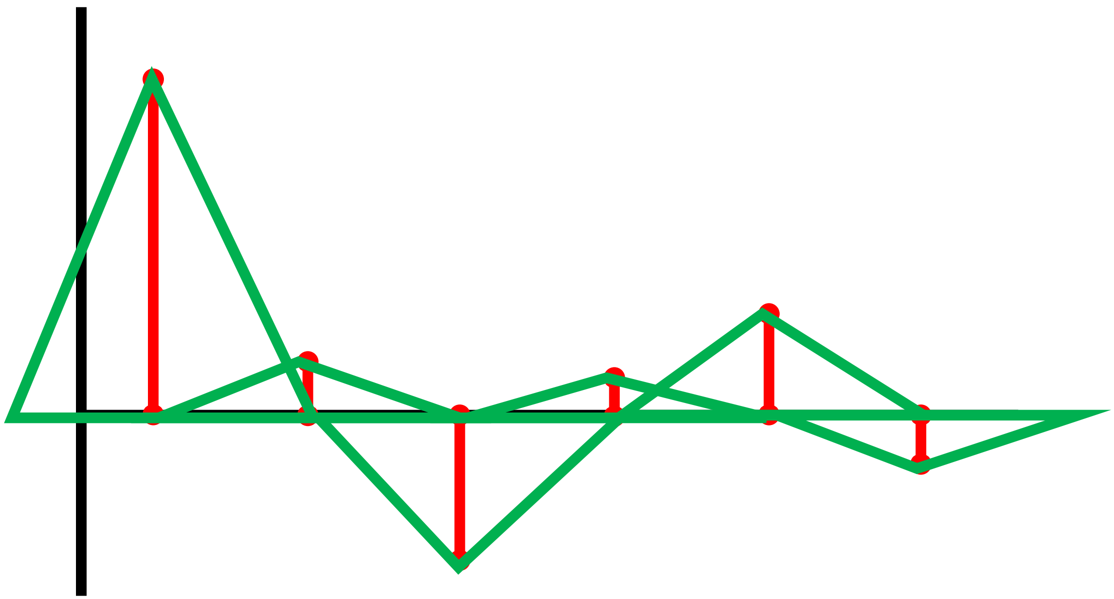
Convolution



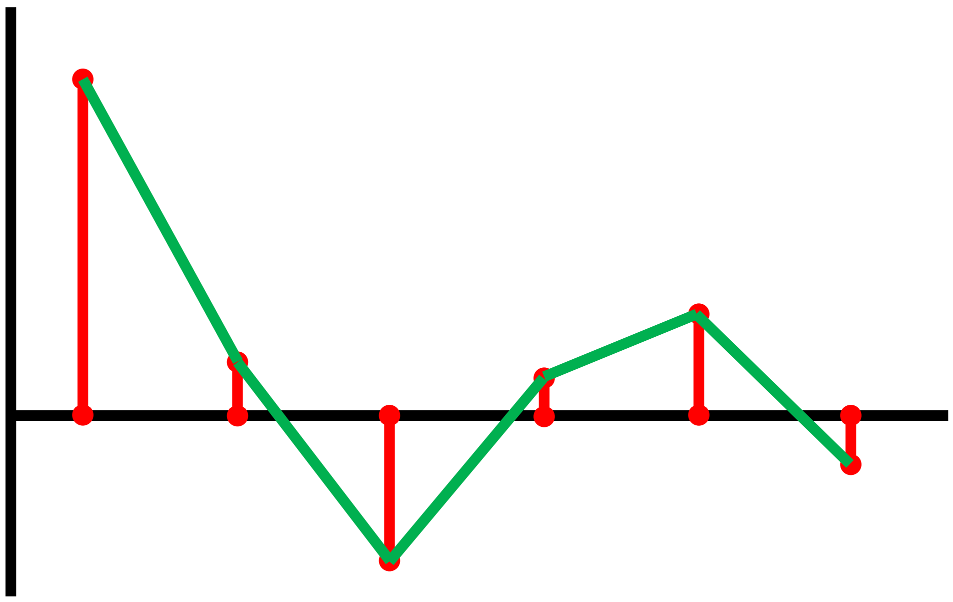
Convolution



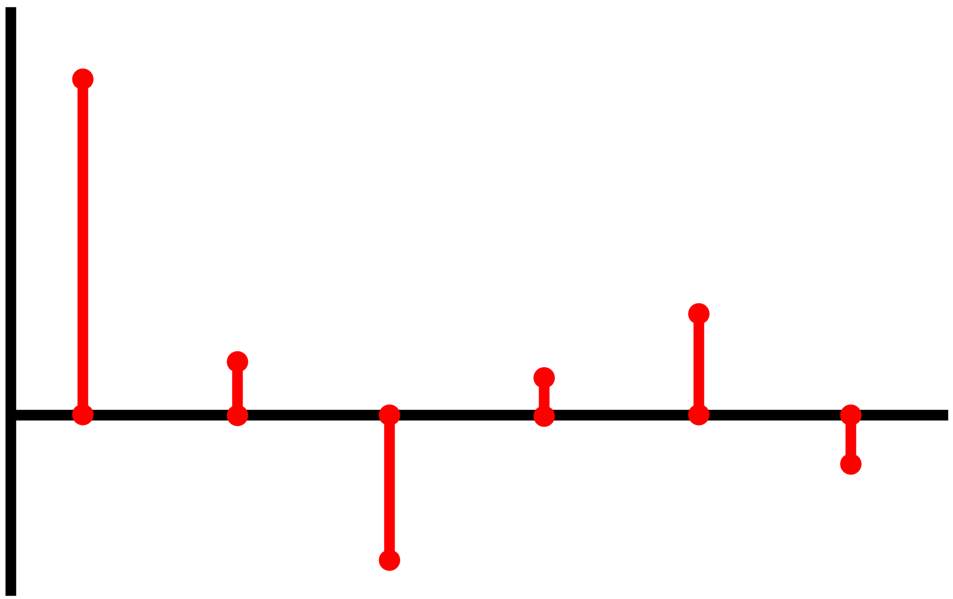
Convolution

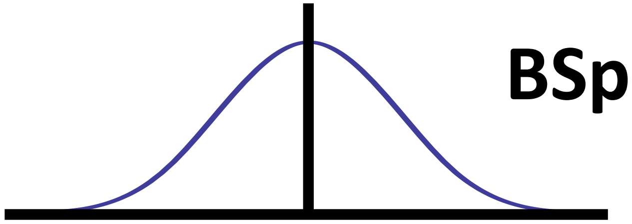


Convolution

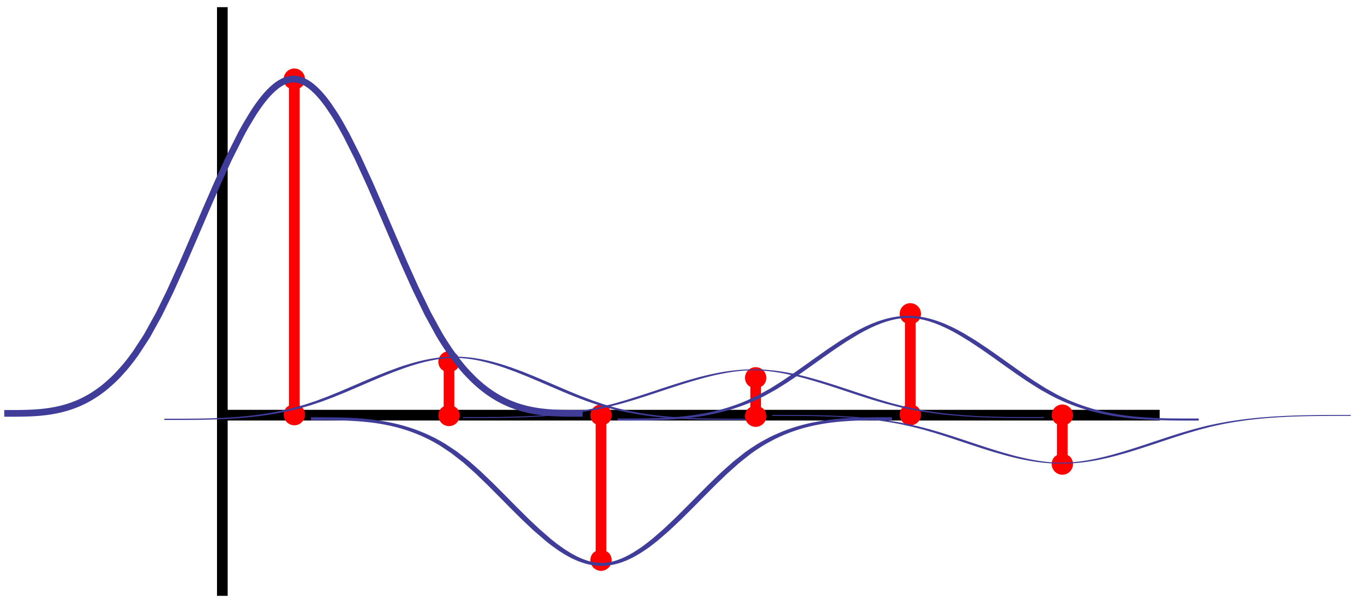


Convolution

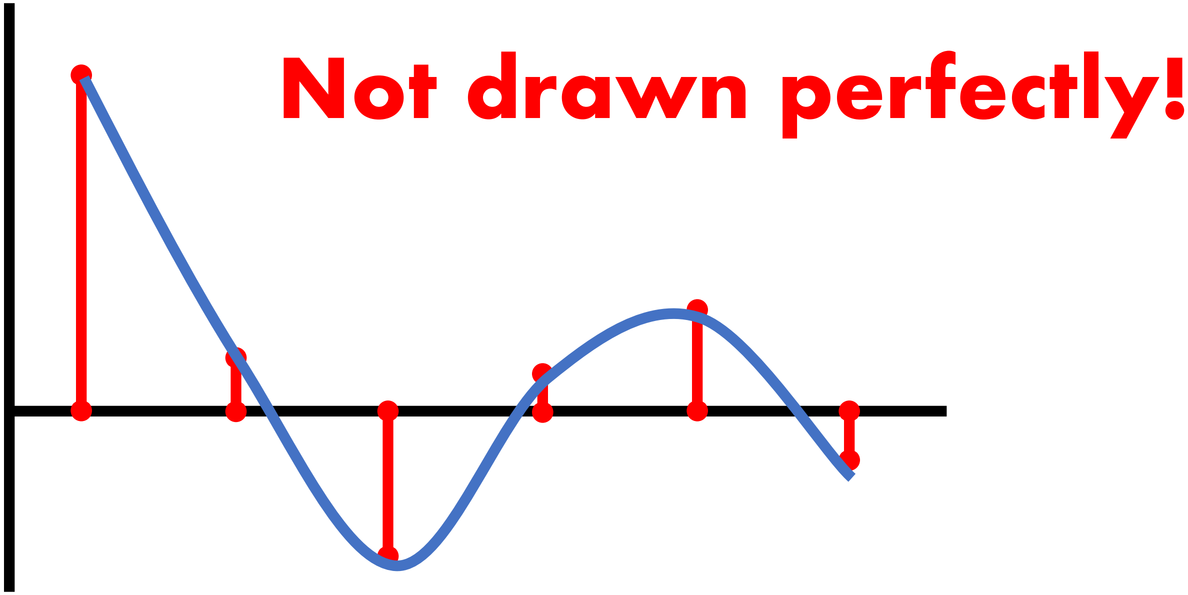




Convolution

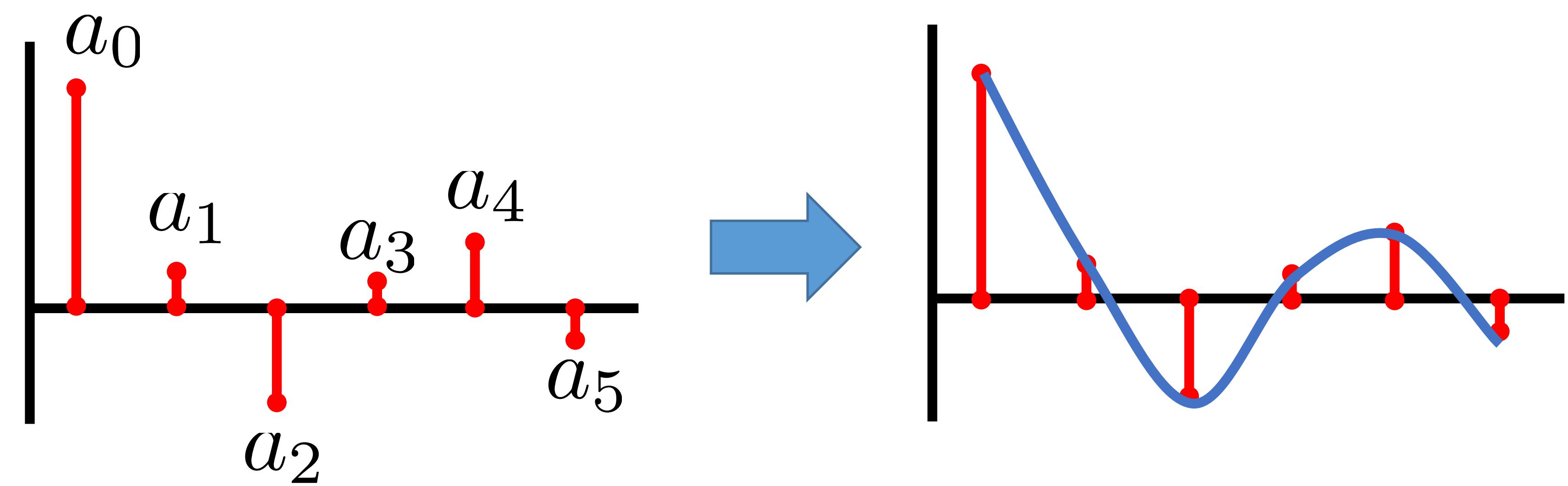


Convolution



Convolution: Avenue to Continuous

Discrete Function **Continuous Function**



$$\{a_0, a_1, a_2 \ldots a_n\}$$

$$\hat{g}(x) = \sum_i a_i k(x - i)$$

Convolution

Convolution of a Discrete and a Continuous Function

$$\hat{g}(x) = \sum_i a_i k(x - i)$$

Convolution

Convolution of a Continuous Functions

$$\hat{g}(x) = \int_u f(u)k(x-u)du$$

Convolution Notations

For continuous function

$$f(x) * k(x) = \int_u f(u)k(x-u)du$$

For discrete function

$$f[i] * k(x) = \sum_i f_i k(x-i)du$$

Convolution Properties

- Convolution is also a linear operator
- Convolution is commutative
  - $H * G = G * H$
  - i.e. if you have two filters (linear) to be applied one after another, the order doesn't matter

Fourier Transform and Convolution

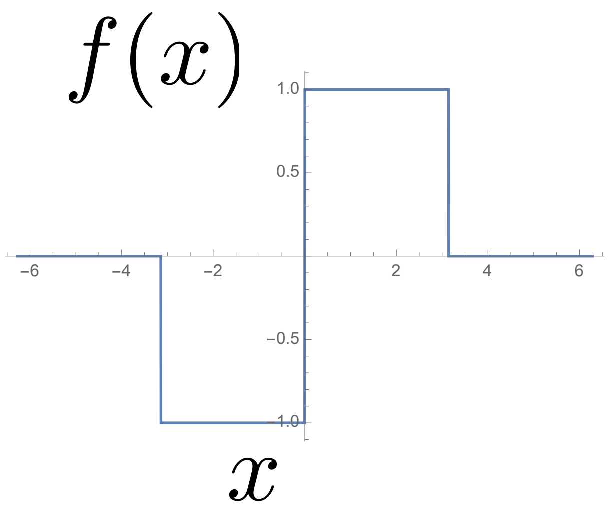
$$g = f * k$$

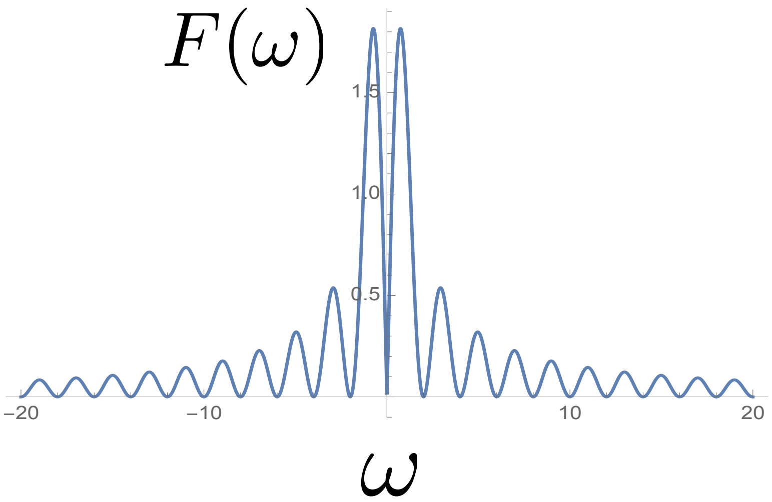
$$\Longrightarrow \mathcal{F}[g] = \mathcal{F}[f] \cdot \mathcal{F}[k]$$

$$G(\omega) = F(\omega) \cdot K(\omega)$$

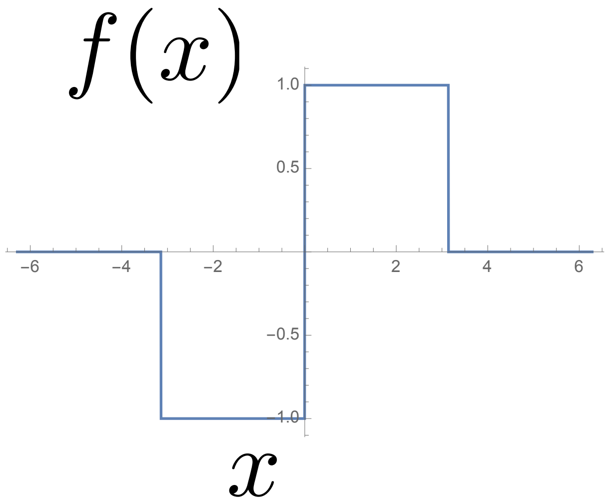
Convolution turns into simple multiplication in the Fourier/Frequency domain

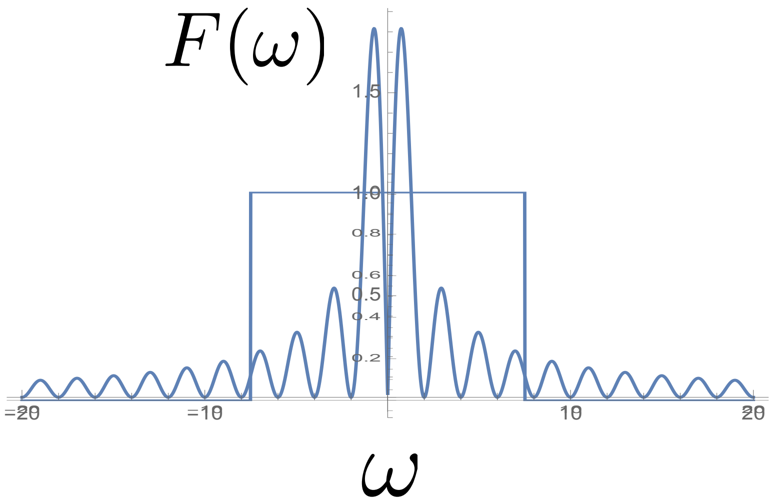
Example



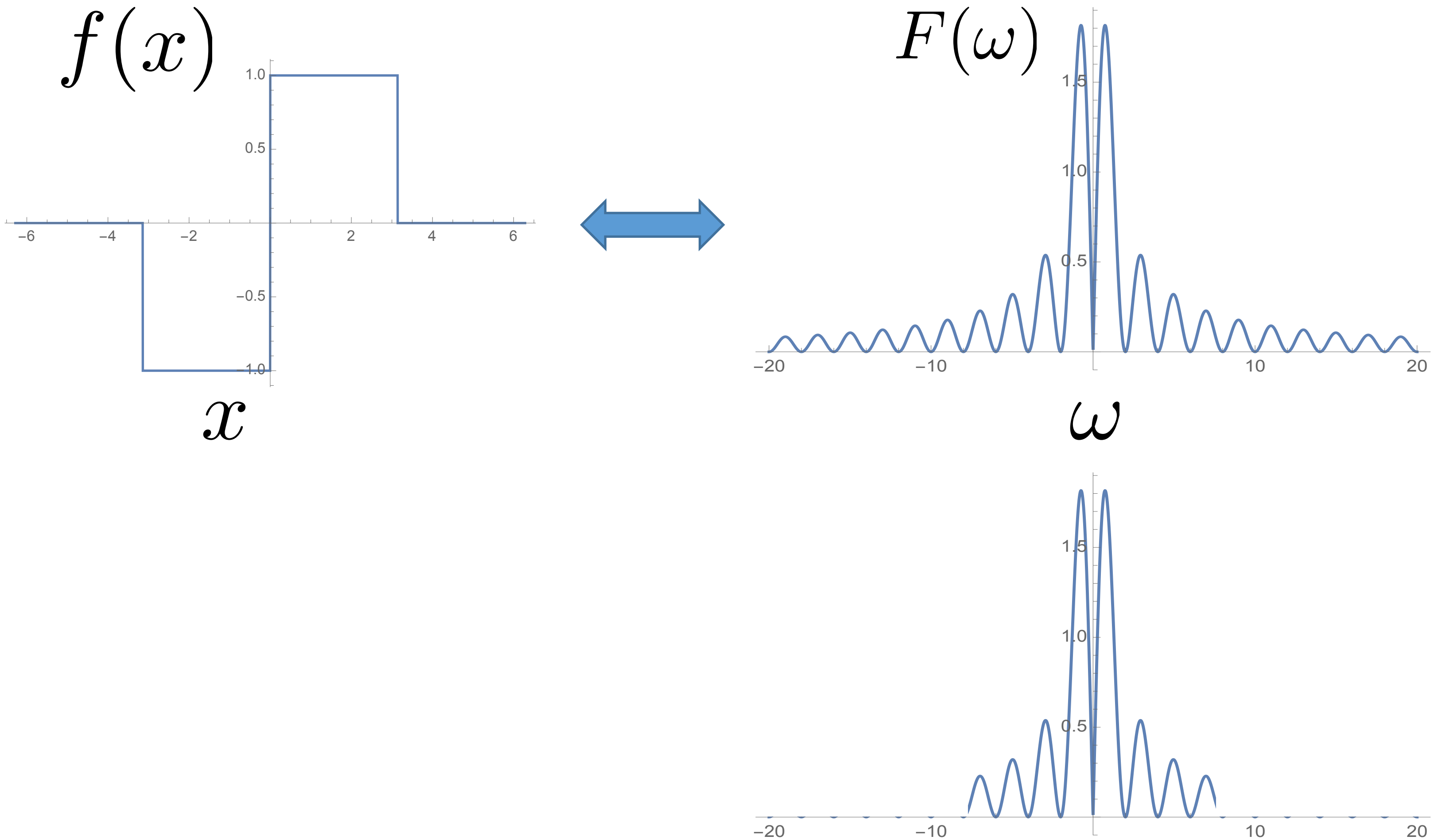


Example

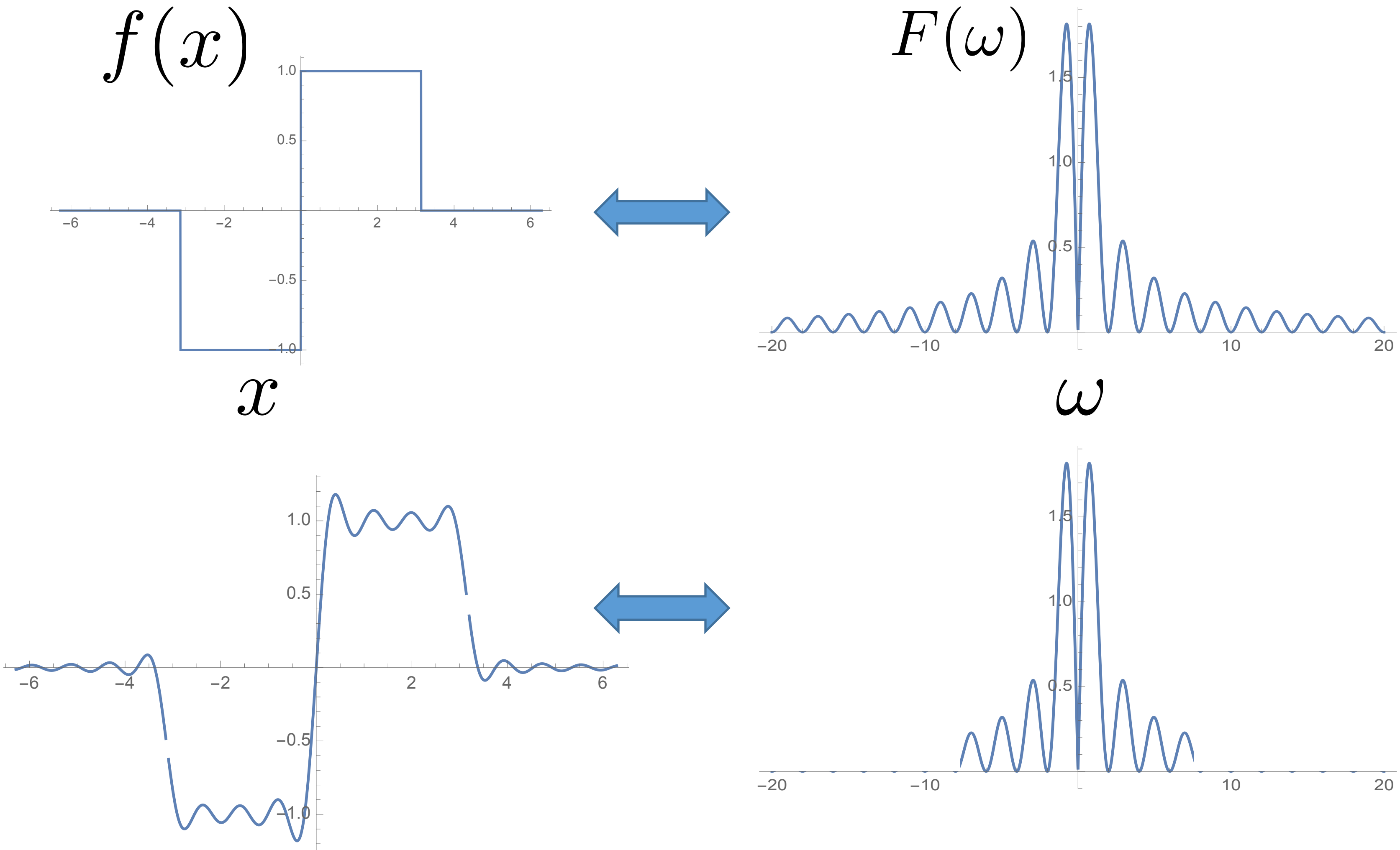

$f(x)$
1.0
0.5
−0.5
−1.0
−6
−4
−2
2
4
6
x


$F(\omega)$
1.5
1.0
0.8
0.6
0.5
0.4
0.2
−20
−10
10
20
ω

Example

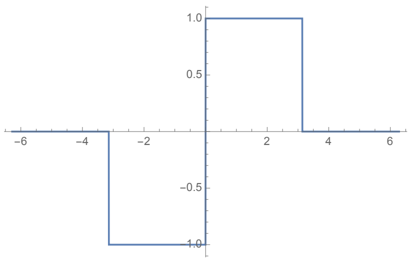

$f(x)$
1.0
0.5
−0.5
−1.0
−6
−4
−2
2
4
6
x
$F(\omega)$
1.5
1.0
0.5
−20
−10
10
20
ω
1.5
1.0
0.5
−20
−10
10
20

Example

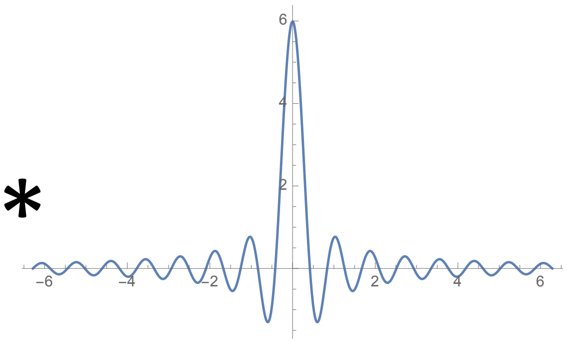

$f(x)$
$F(\omega)$
1.0
0.5
−0.5
−1.0
−6 −4 −2 2 4 6
1.5
1.0
0.5
−20 −10 10 20
x
ω
1.0
0.5
−0.5
−1.0
−6 −4 −2 2 4 6
1.5
1.0
0.5
−20 −10 10 20

Example

Spatial Domain



*



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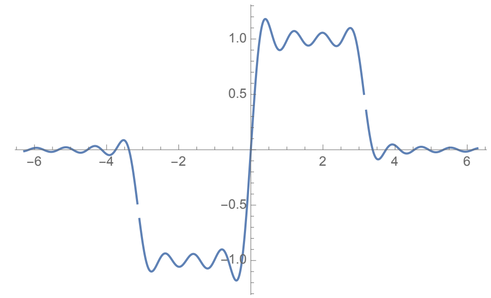
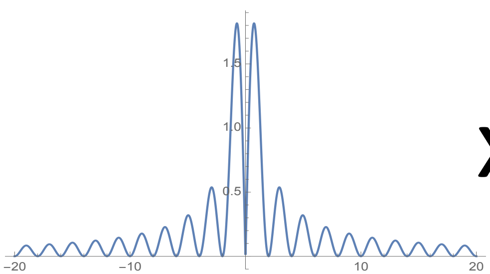
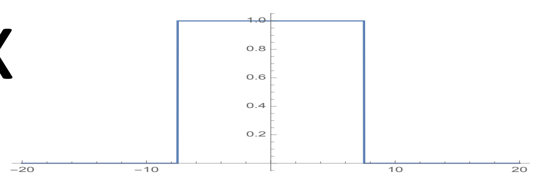


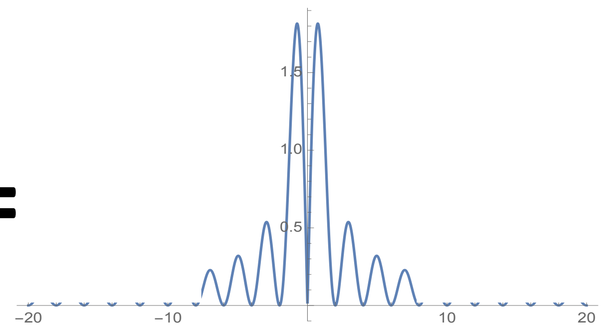
Frequency Domain



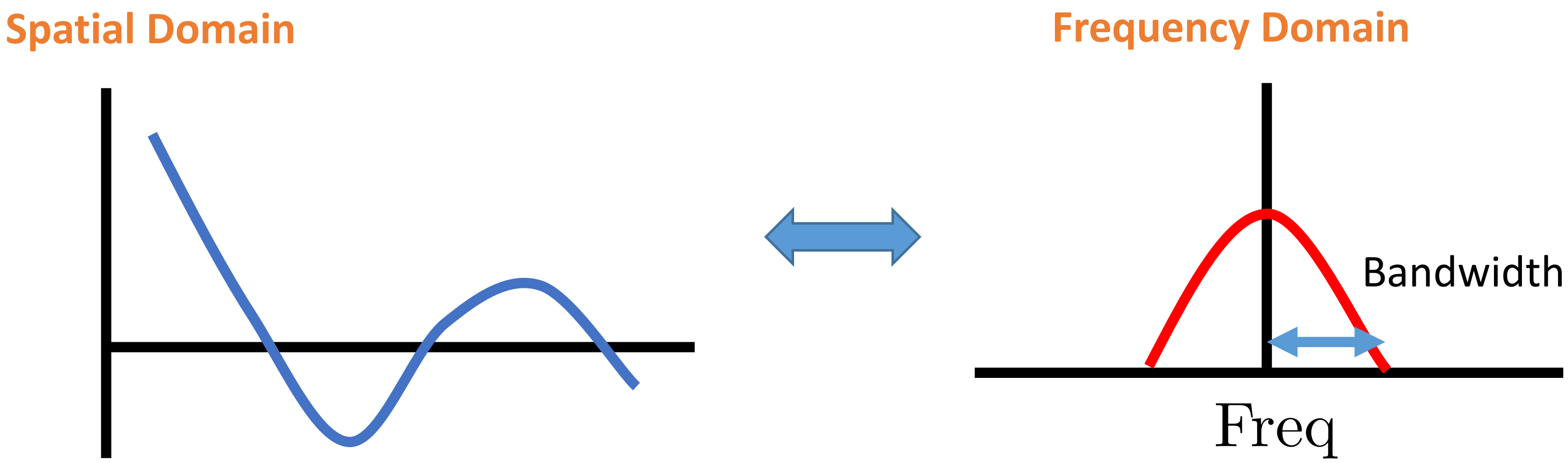
X



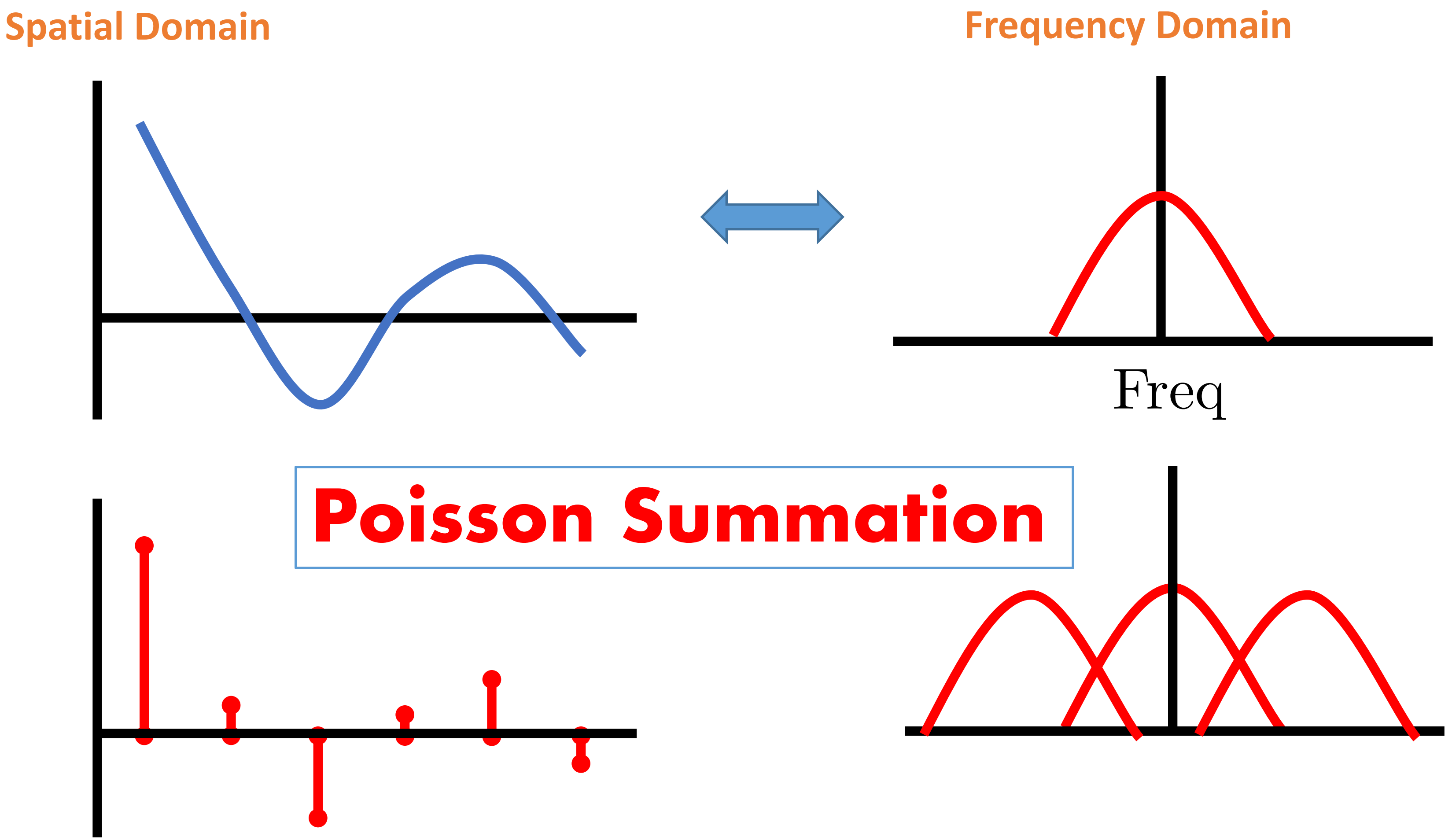
=



Fourier Transform of Discrete Function



Fourier Transform of Discrete Function



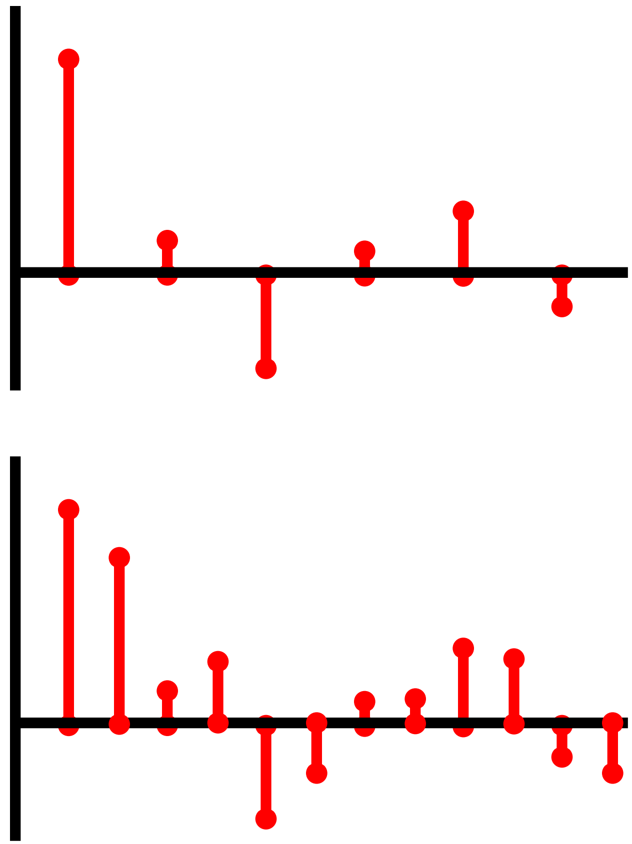
Poisson Summation

$$\mathcal{F}\left[f(nT)\right] = \sum_k F\left(f - k\frac{1}{T}\right), \quad \mathcal{F}\left[f(x)\right] = F(f), \text{sampling rate} = \frac{1}{T}$$

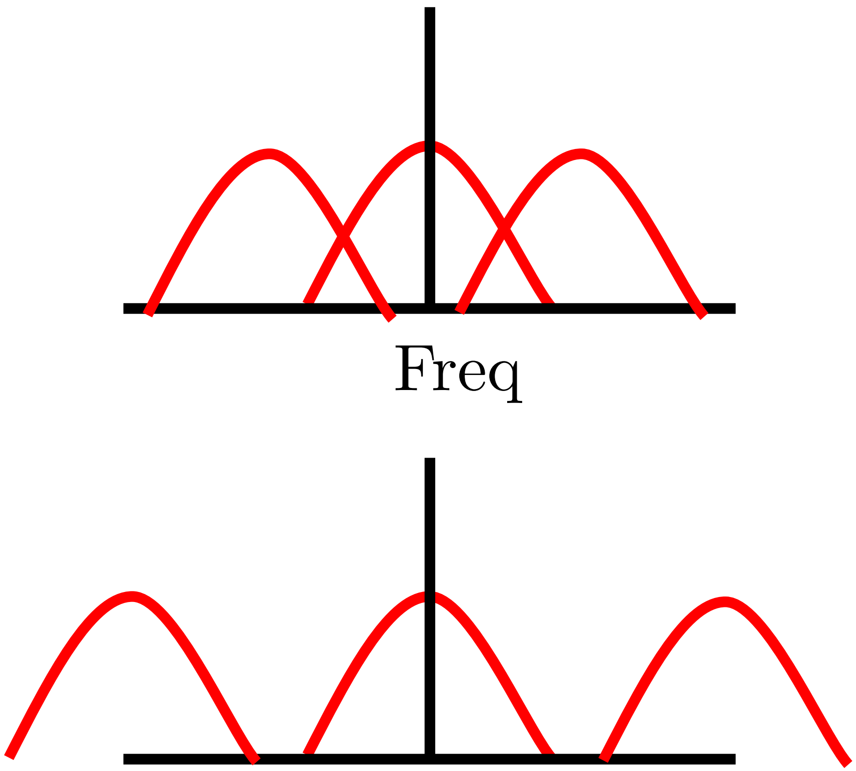
Note the use of frequency variable f, instead of w

Sampling Rate

Spatial Domain

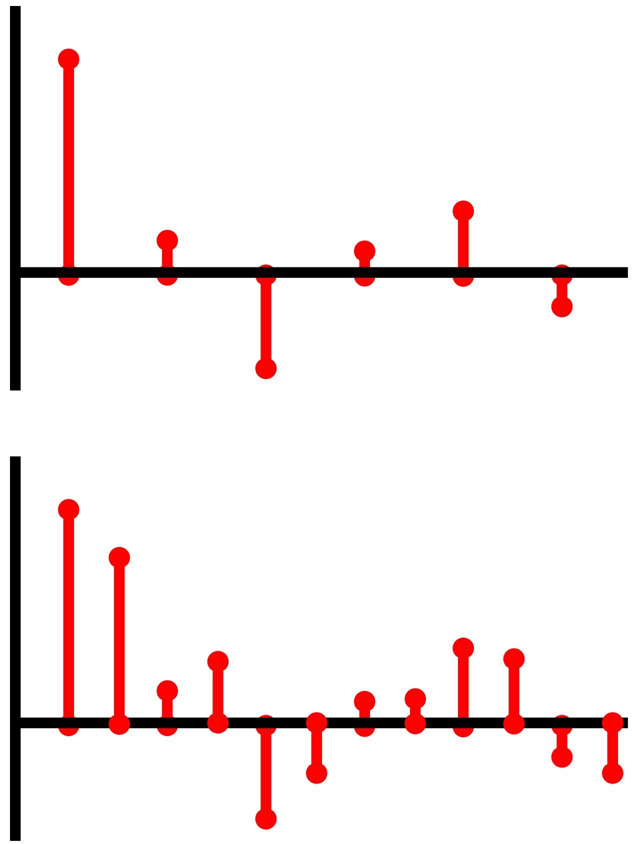


Frequency Domain

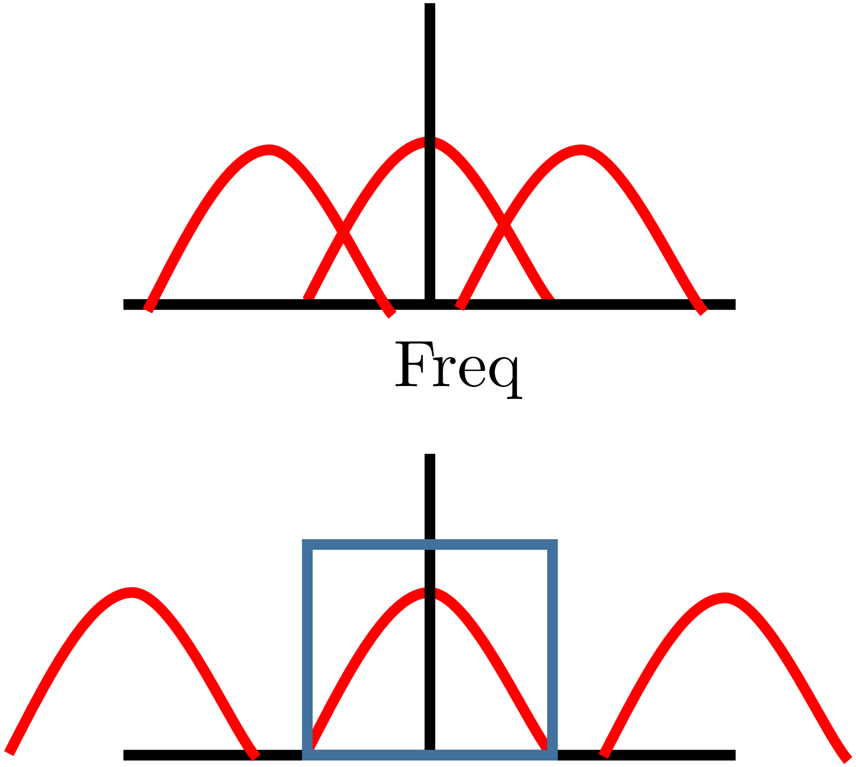


Sampling Rate

Spatial Domain



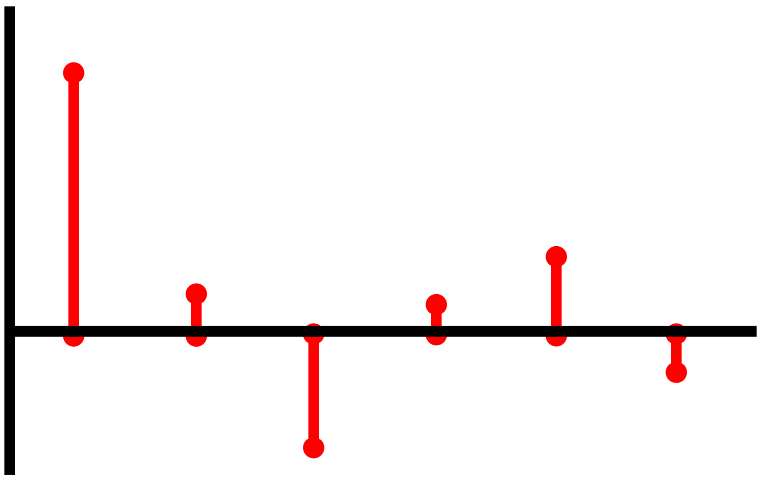
Frequency Domain



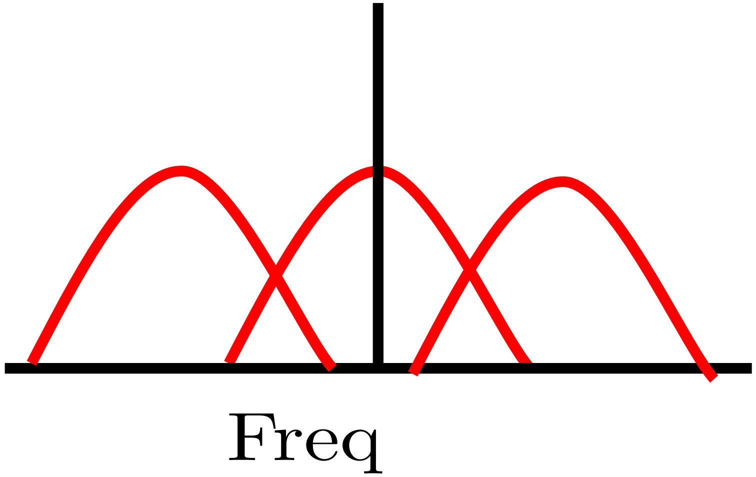
Can be perfectly reconstructed !

Aliasing

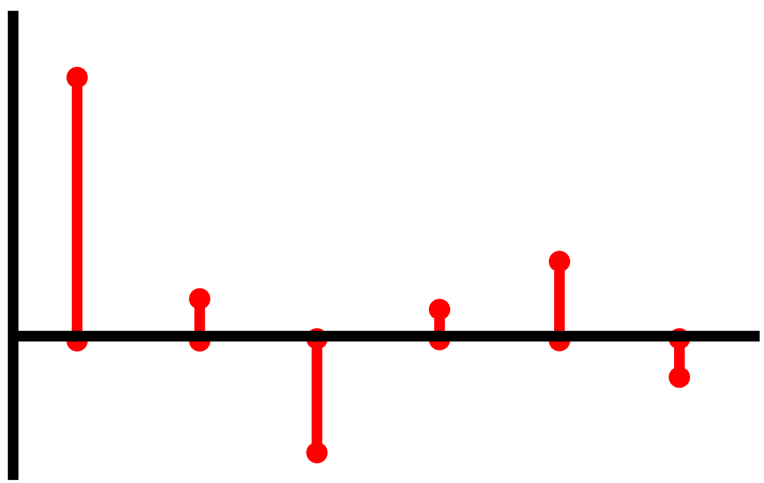
Spatial Domain

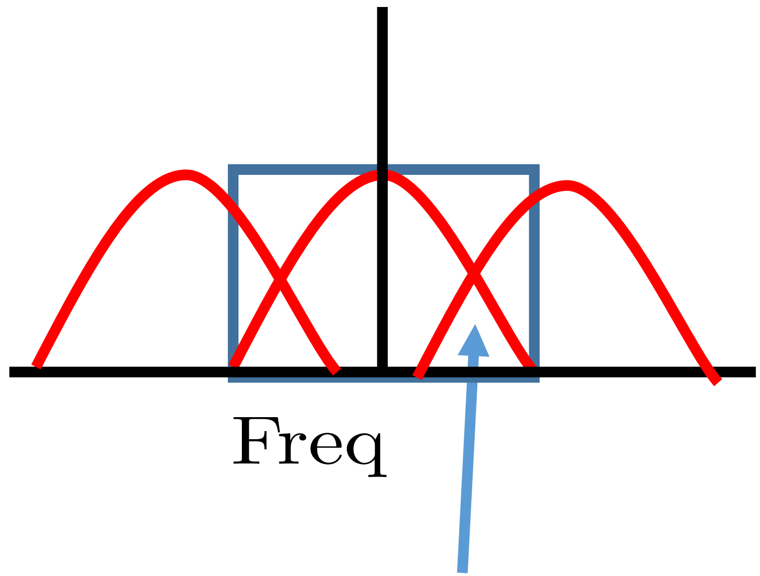


Frequency Domain



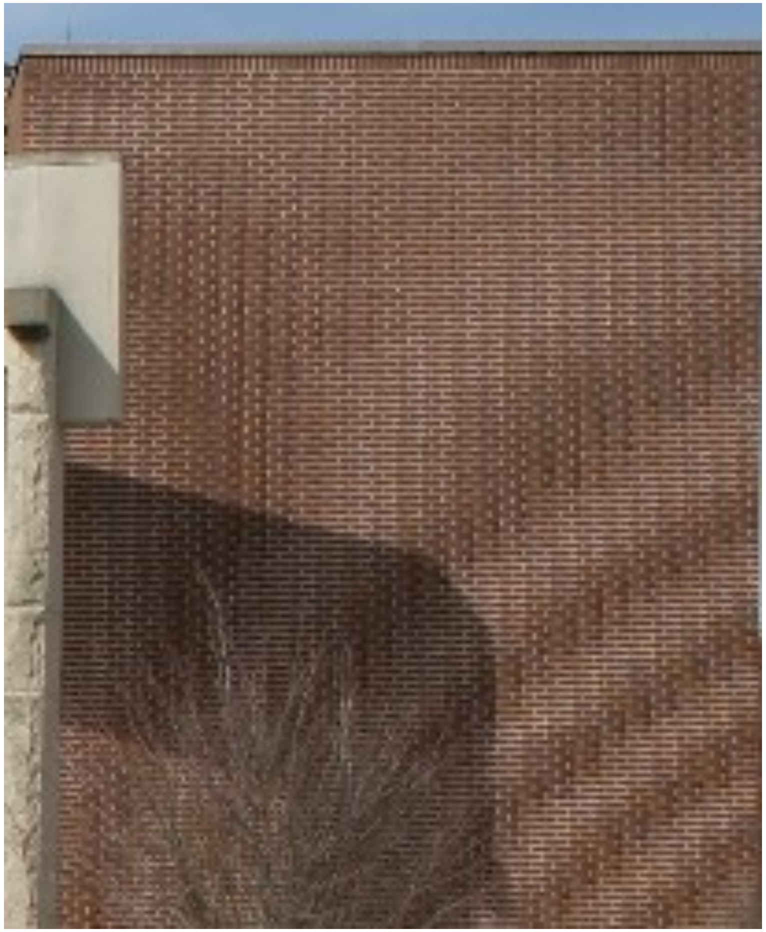
Aliasing





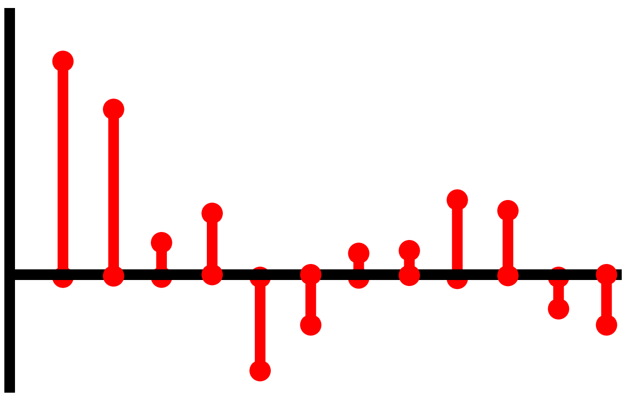
Cannot be reconstructed perfectly !
High frequencies appear as low frequencies.

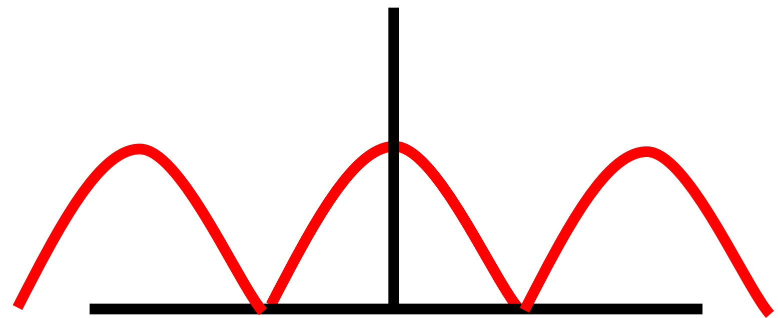
Aliasing: Moiré Pattern



High frequencies appearing as low frequencies !

Optimal Sampling





Copies of F(f) are just touching

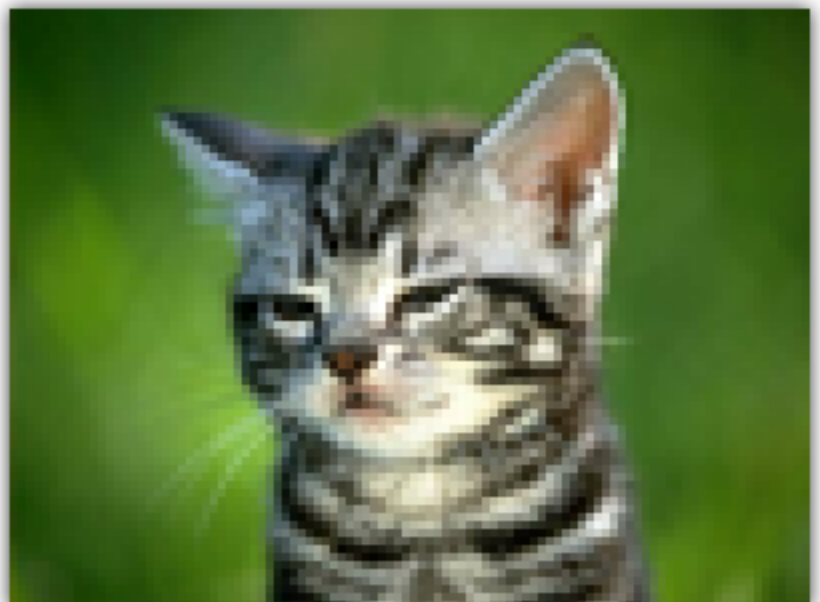
$$\frac{1}{T} = \text{Sampling Rate} = 2 \times \text{Bandwidth}$$

Nyquist Citerion/Theorem

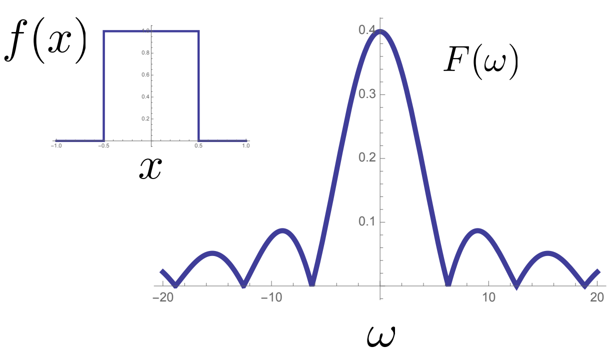
Example

- Human voice bandwidth is 4Khz
- Sampling rate must be atleast 8Khz
- Each sample is stored with 8 bits
- Therefore voice encoding requires 64Kbps (PCM)

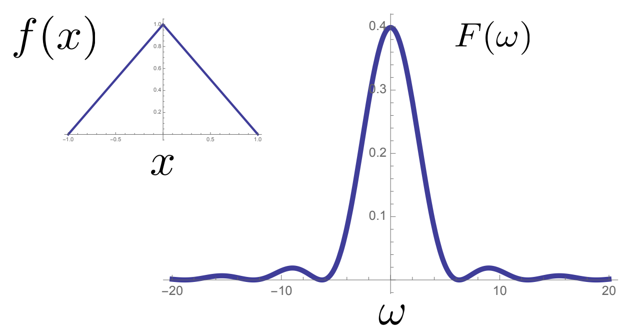
OpenGL Filtering (Fourier Domain)



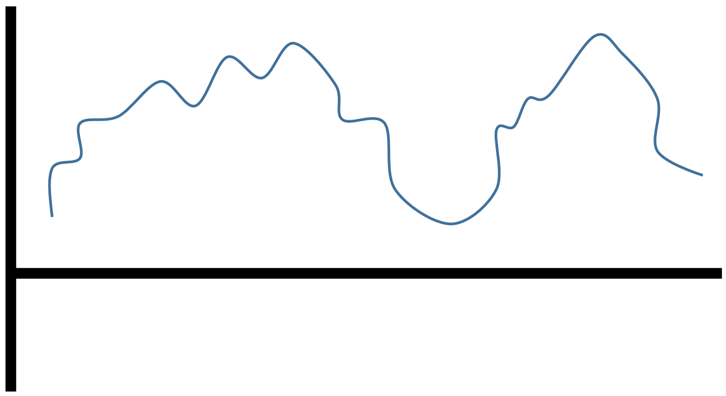
GL_NEAREST

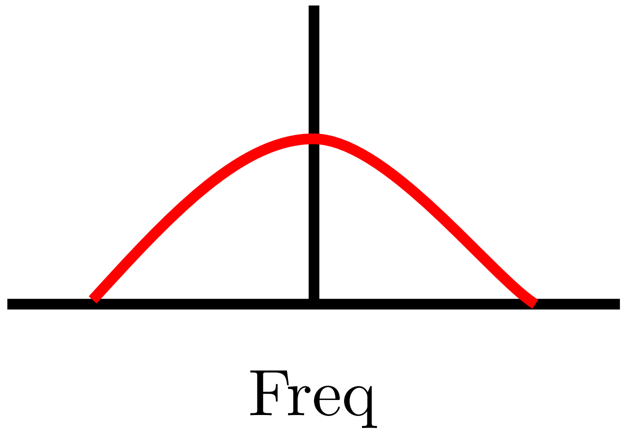


GL_LINEAR

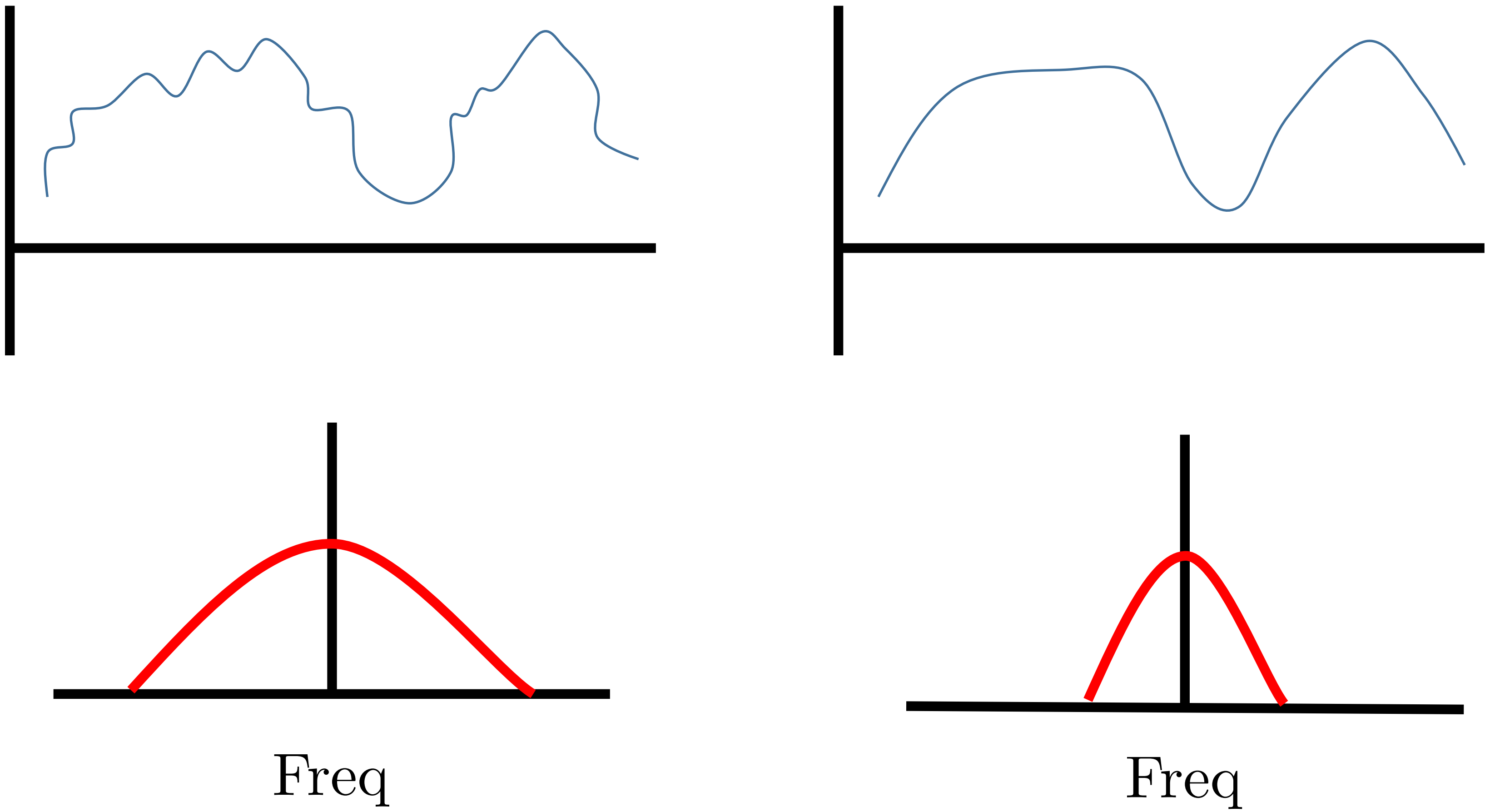


Sampling/Subsampling



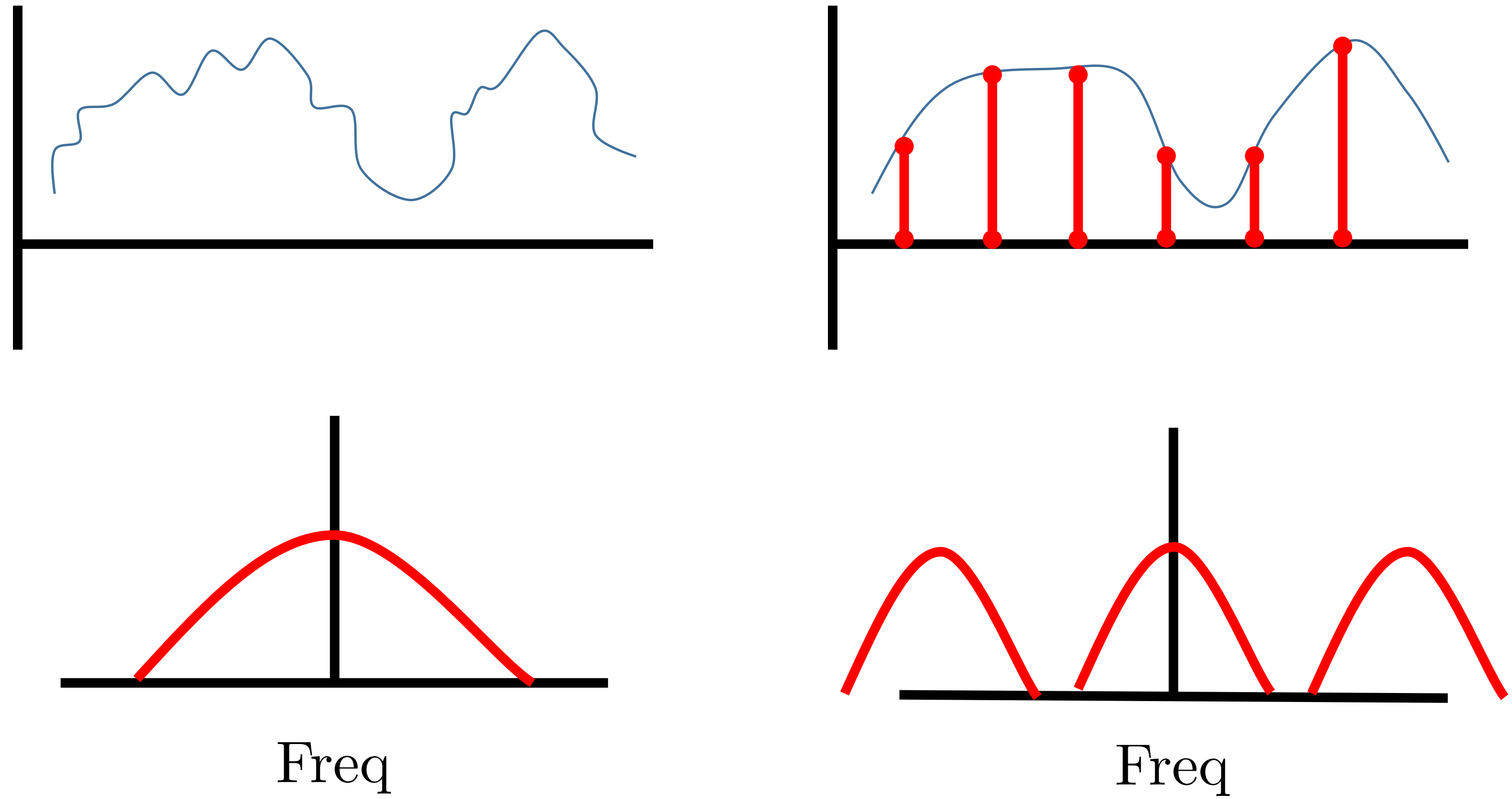


Sampling/Subsampling



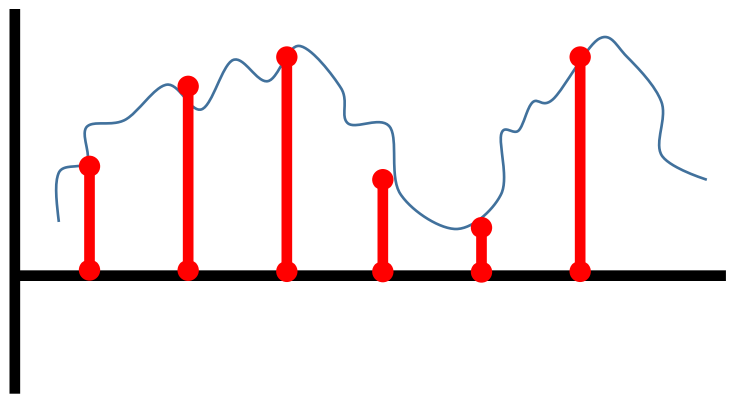
Low-pass filter first: remove high frequencies

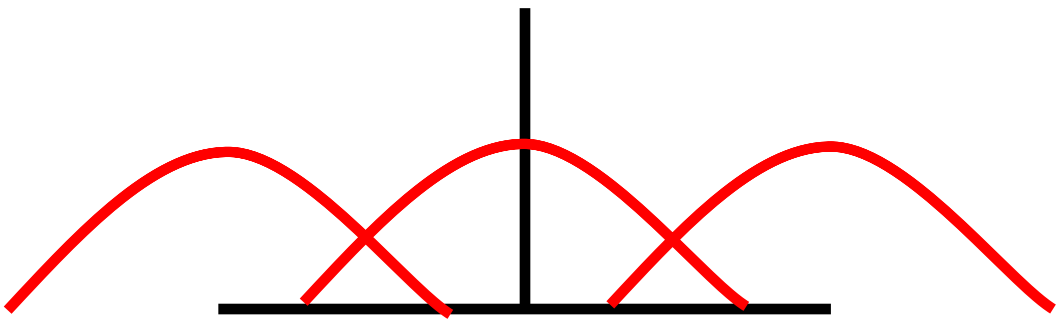
Sampling/Subsampling



Sample after low-pass filtering: no aliasing

Sampling/Subsampling



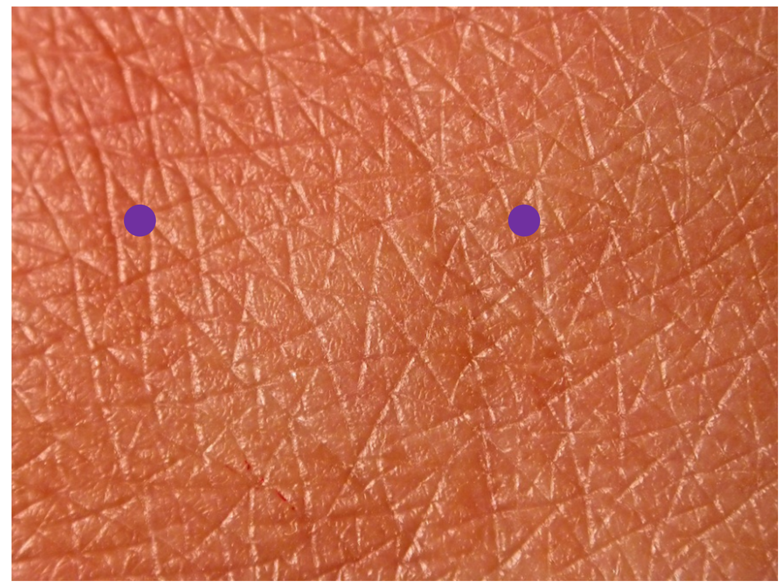


Sample before low-pass filtering: aliasing !

Example: Subsampling Textures

Small image

Large texture



Subsampling: Filtering during sub-sampling inherently does anti-aliasing
But the filter size needs to much larger

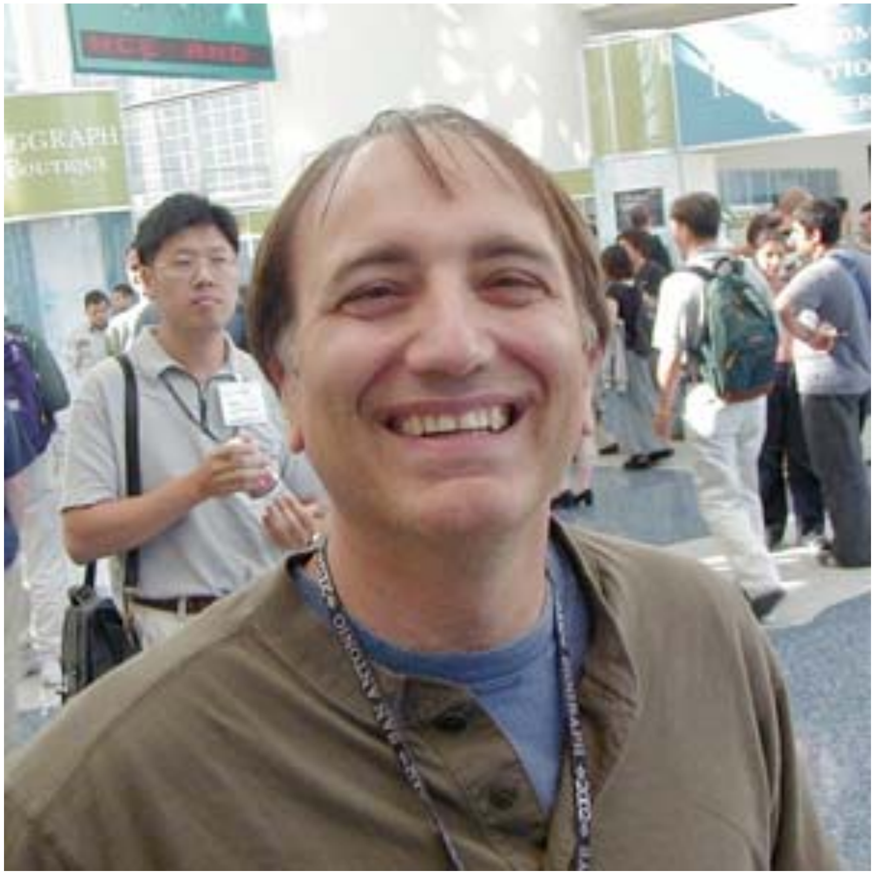
Revisit Mipmapping

- Progressively anti-alias and sub-sample
- Pre-computes all the levels
  - Thus avoids large filter during run-time

FFT: Fast Fourier Transform

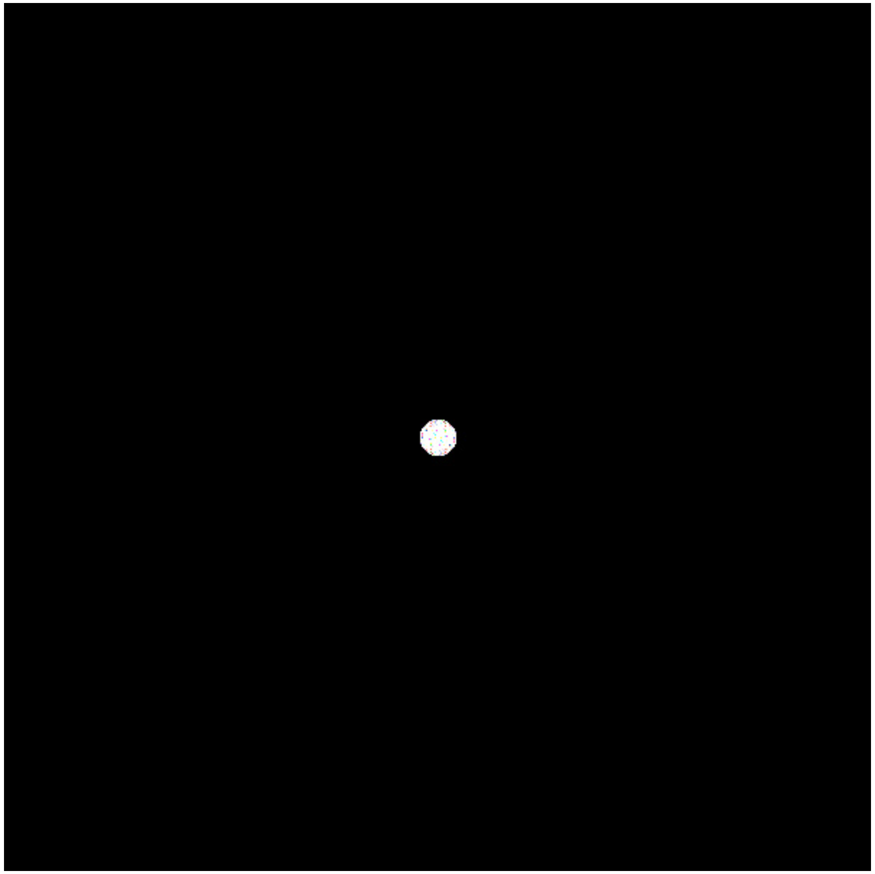
- Discrete Fourier Transform
- A very efficient method to compute discrete fourier transform.
- Extremely important algorithm !

Real World Images



Spatial Domain | Frequency Domain

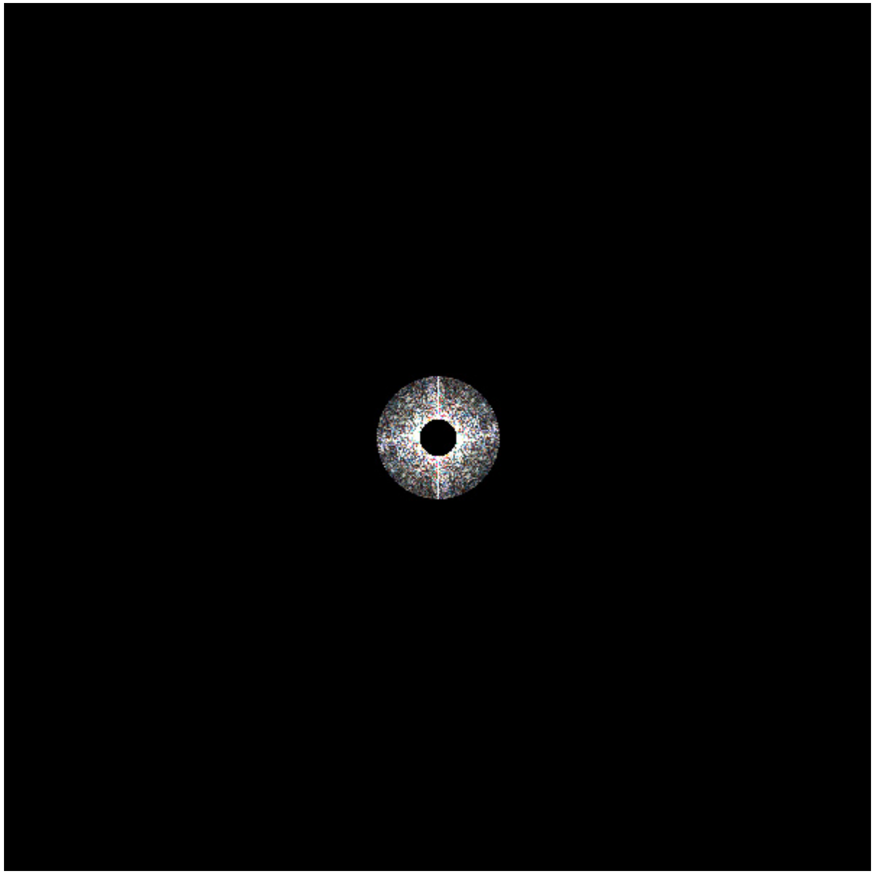
Low-Pass Filter



Spatial Domain

Frequency Domain

Band-Pass Filter

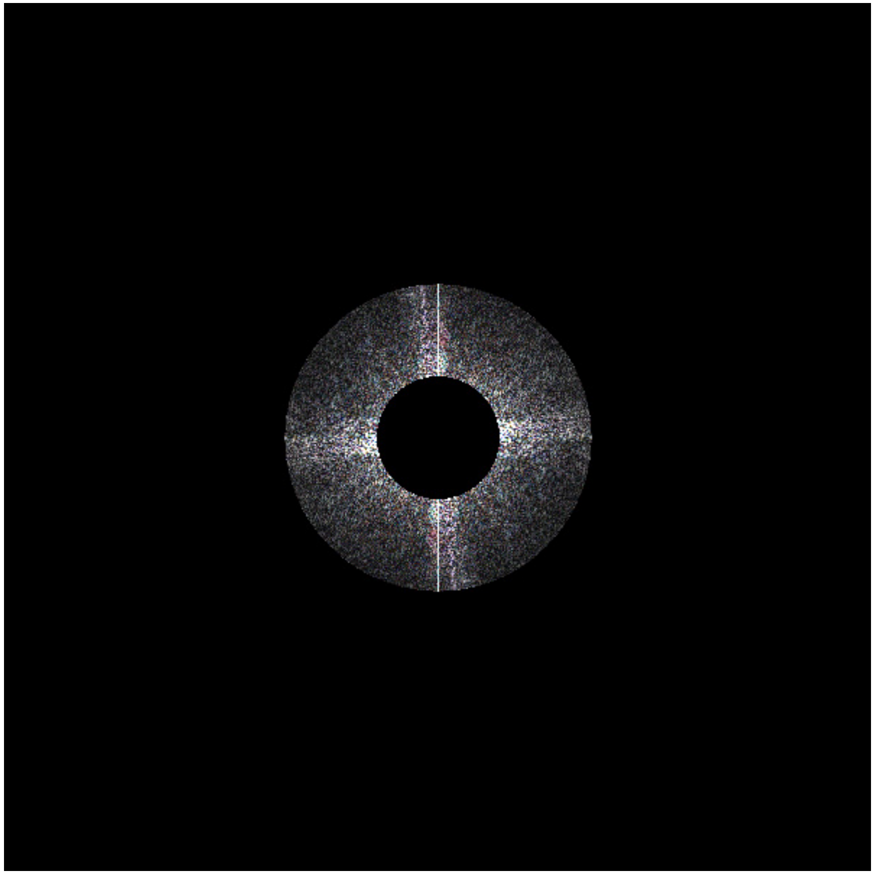


Spatial Domain

Frequency Domain

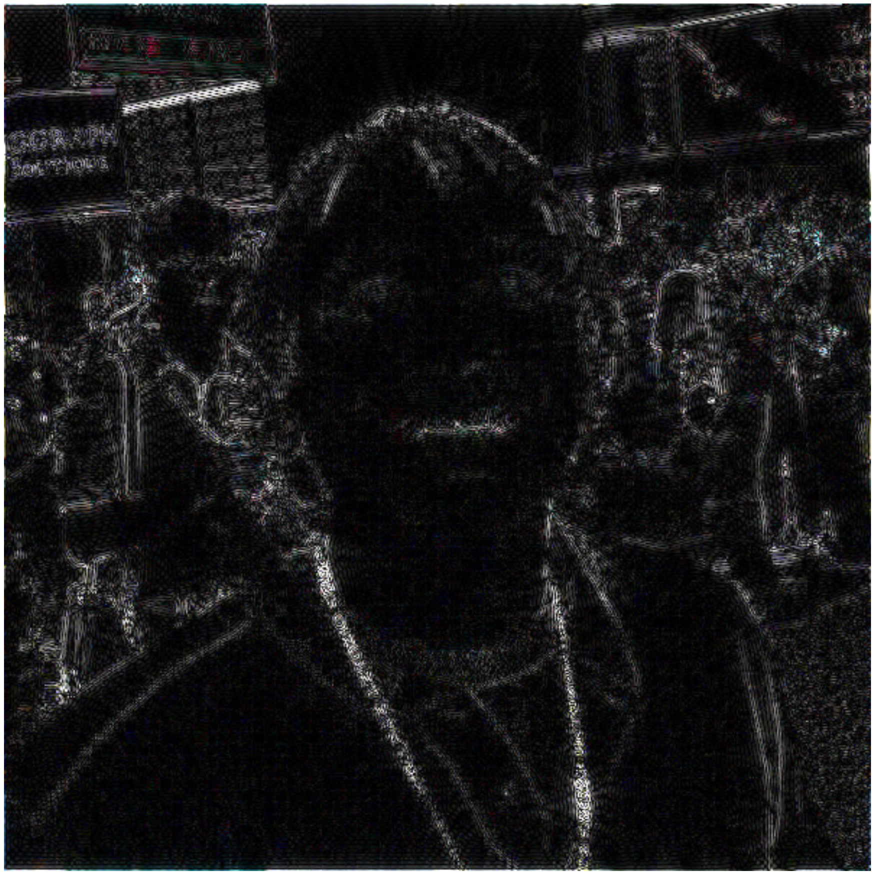
Larger Band-Pass Filter

Spatial Domain

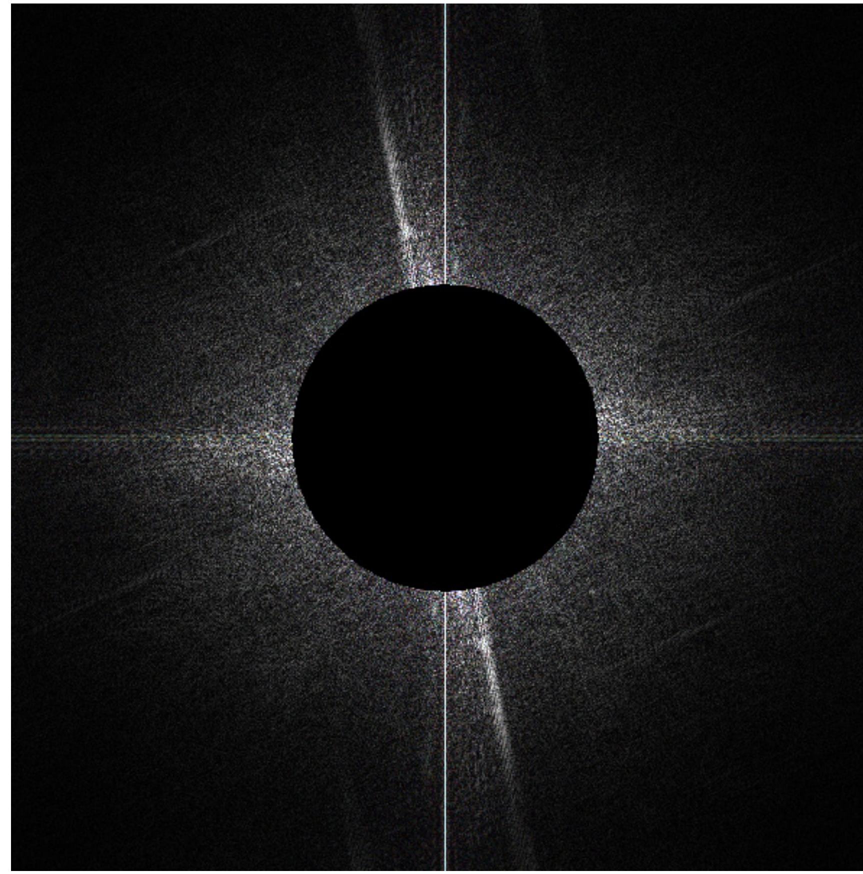


Frequency Domain

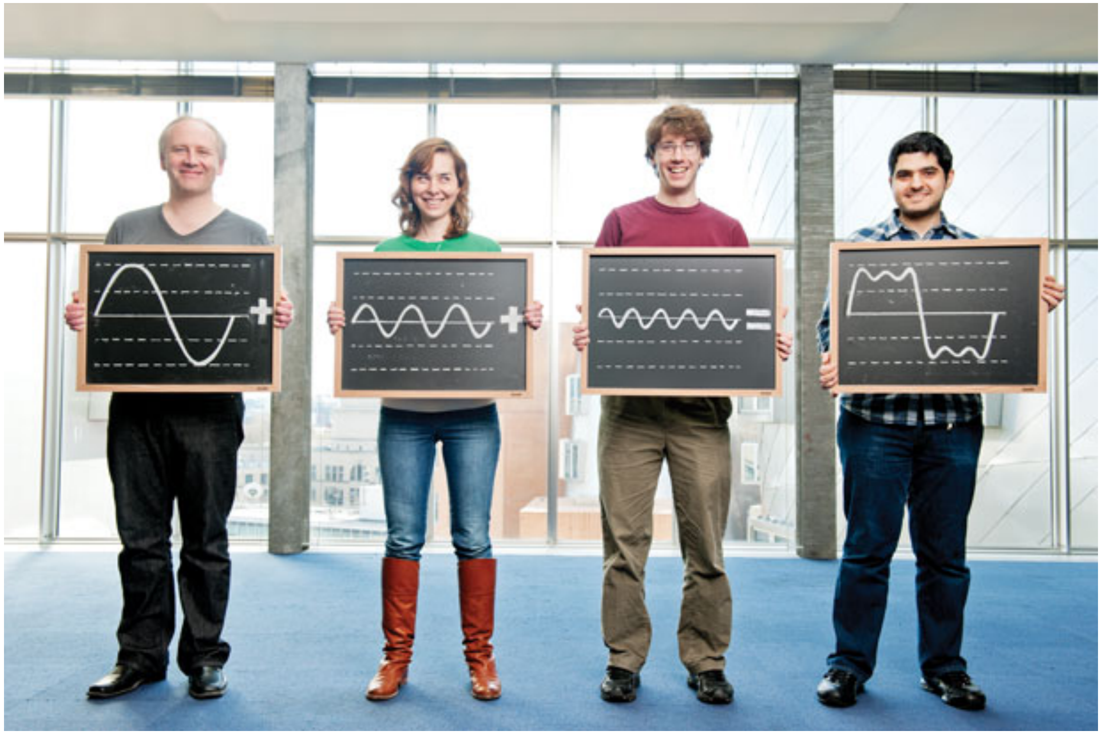
High-Pass Filter (Edges)



Spatial Domain



Frequency Domain



Sampling

CS 148: Summer 2016
Introduction of Graphics and Imaging
Zahid Hossain