Homework 1: Rasterization and Transformation

CS 148: Introduction to Computer Graphics and Imaging (Summer 2016)
Stanford University

We derived Bresenham's algorithm for rasterizing lines in class. In this assignment, we are going to derive and implement a similar algorithm for rasterizing circles. In particular, we will rasterize the quadrant of the circle $x^2 + y^2 = R^2$ where $x, y \geq 0$ and positive integer R.

We suggest you to read "Computer Graphics Principles and Practice - Foley et. al" section 3.2 "Scan Converting Lines" and section 3.3 "Scan Converting Circles." It should be included in the handout as reading.pdf.

One obvious way you could attempt to rasterize such a circle would be to iterate over each x corresponding to a pixel center and light up the pixel closest to $(x, \sqrt{R^2 - x^2})$.

Problem 1. *Why is this approach bad? Provide at least two reasons.*

Instead, we will imitate our development of Bresenham's algorithm for rasterizing lines. In fact, we can simplify matters somewhat: We can employ the symmetry of the circle to reduce the number of pixels we have to rasterize. Then, each time we rasterize a pixel we will also light up its symmetric counterpart.

Problem 2. *Suppose we are given (x, y) on the 45° arc of the circle between $(0, R)$ and $(R/\sqrt{2}, R/\sqrt{2})$. Find a corresponding symmetric point on the 45° arc from $(R/\sqrt{2}, R/\sqrt{2})$ to $(R, 0)$ that takes no additional operations to compute.*

For deriving methods like Bresenham's algorithm, it is often more convenient to use *implicit* forms of curves. So, we define $F(x, y) = x^2 + y^2 - R^2$. Then, our curve is given by $F(x, y) = 0$. Note that $F(x, y) < 0$ when (x, y) is inside the circle and $F(x, y) > 0$ when (x, y) is outside the circle.

Just as in our method for rasterizing lines, we will be marching from left to right. Let's say we are at a pixel whose center is (x, y). Then, neighboring pixel centers are given by $(x \pm 1, y \pm 1)$. Our algorithm will make a local decision about which neighbor to choose next in its walk around the circle.

Problem 3. *Show that the slope of the circle in the 45° arc we are drawing satisfies $-1 \leq dy/dx \leq 0$. Use this fact to justify why if we trace pixels in increasing x along the circle, it is sufficient to consider only the neighbors $(x + 1, y)$ and $(x + 1, y - 1)$ of (x, y).*

Suppose we just drew the pixel at (x_P, y_P). Our method will have the property that the circle will intersect the segment between $(x_P, y_P - 1/2)$ and $(x_P, y_P + 1/2)$ (it may help if you draw this for yourself). We will define our *decision variable* to be $d = F(x_P + 1, y_P - 1/2)$.

Problem 4. *Give a rule using d for choosing between $(x_P + 1, y_P)$ and $(x_P + 1, y_P - 1)$ using the property explained above.*

We now have a rule for choosing which pixel to visit next, but now we'll need to update d. We will call the move from (x_P, y_P) to $(x_P + 1, y_P)$ a move *east* (E) and the move from (x_P, y_P) to $(x_P + 1, y_P - 1)$ a move *southeast* (SE).

Problem 5. *Provide update rules of the form $d_{new} = d_{old} + \Delta_E$ and $d_{new} = d_{old} + \Delta_{SE}$ to be applied depending on which pixel you choose.*

Excitingly, you should find that if x_P, y_P are integers, so are Δ_E and Δ_{SE}. This suggests that we can do integer arithmetic while rasterizing circles, which can buy us some time!

To complete our rasterization algorithm, we just need a way to start out. For simplicity, we'll actually change the convention we explained in class and assume *pixel centers coincide with integer coordinates*. So, we'll start at $(x, y) = (0, R)$.

Problem 6. *Provide an initial value for d and show that it isn't an integer. Suggest an alternative decision variable h of the form $h = d - c$ for some fixed constant c so that the initial value of h is an integer, and provide update and decision rules for h.*

Problem 7. *What kind of transformation is the following matrix ? - describe it precisely for full point. What does the eigen vector, with non-zero eigenvalue, of this matrix correspond to ? (You don't need to do any algebra for this - simply state).*

$$\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Problem 8. *Write down the projection of a vector $\vec{v}$ on another vector $\vec{u}$.*

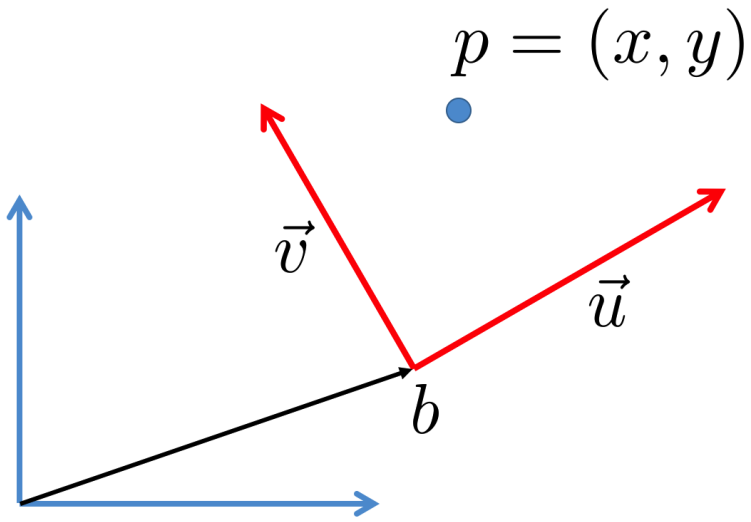

Figure 1: The red frame depicts a 2D camera with position b and basis vectors $\vec{u}, \vec{v}$. Consider the point p is given in global coordinate, i.e. with respect to the blue frame located a $(0, 0)$ with canonical basis.

Problem 9. *Consider a 2D camera frame, as shown in red in Figure 1, with a position b and basis vectors $\vec{u}, \vec{v}$. This camera is looking at a point p that is given in the global coordinate (x, y) (w.r.t the blue frame). In the lecture we have learned that p could be written in the homogeneous coordinate system as the following:*

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} u_x & v_x & b_x \\ u_y & v_y & b_y \\ 0 & 0 & 1 \end{bmatrix}}_{M} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \quad \text{where } p' = (x', y') : \text{ is the coordinate in the local camera frame}$$

Now, one could take the inverse of M to compute p', but we could also do the same much more easily using projections by answering the following short questions:

1. *Write down the vector that point to p with respect to the camera position b.*
2. *Write down the projection of the above vector on $\vec{u}$ and $\vec{v}$ separately (Assume: $|\vec{u}|, |\vec{v}| = 1$)*
3. *What are these projections in terms of the camera frame and the local coordinate p'? Write these projections in form a homogenous transformation.*
4. *If the above transformation allows you to compute p' given p then you must have computed the inverse of M. What special properties of the basis vectors $\vec{u}, \vec{v}$ have allowed you to compute this inverse using only projections ?*

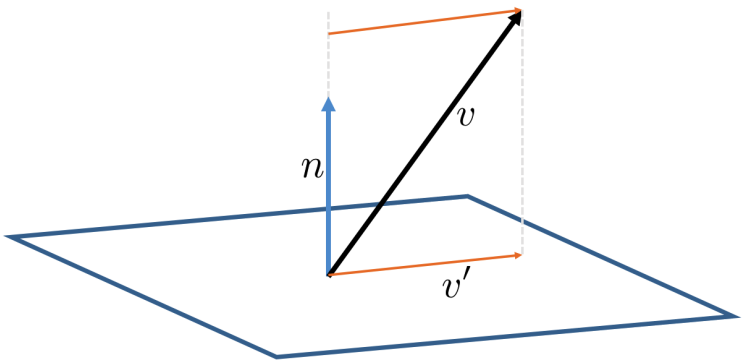

Figure 2: n is the normal vector of the blue plane., i.e its perpendicular to the plane and $|n| = 1$. v' is the projection of the vector v on the plane.

Problem 10. *Referring to Figure 2: consider a plane with normal n, $|n| = 1$. Show that the projection v' of a vector v on the normal plane (defined by the normal n) is given by.*

$$v' = \left(I - nn^T\right) v$$

(Hint: A dot product between two column vectors a and b can be written as matrix multiplication, i.e. $a \cdot b = a^T b = b^T a$.)

Problem 11 (Extra Credit). *Rasterize an ellipse that is described the following equation:*

$$b^2x^2 + a^2y^2 = a^2b^2$$

You can assume that $a > b$, thus $-a \ldots a$ is the major axis and $-b \ldots b$ is the minor axis. Rasterizing only one quadrant is sufficient for the purpose of this extra credit problem.

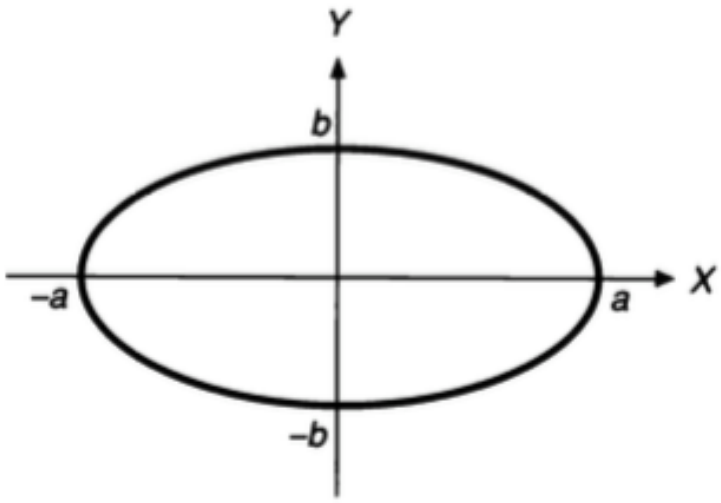

Figure 3: An ellipse with $-a \ldots a$ as the major axis and $-b \ldots b$ the minor axis